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XII.

FURTHER DEVELOPMENTS OF THE GEOMETRICAL THEORY OF SIX SCREWS.

By SIR ROBERT BALL, LL.D., F.R.S., Lowndean Professor of Astronomy and Geometry in the University of Cambridge.

[Read February 11, 1901.]

Since the publication in 1897 of the last memoir on the "Theory of Screws" that I had the honour of laying before the Academy, the entire subject, so far as I had then been able to become acquainted with it, has been incorporated in a volume entitled "A Treatise on the Theory of Screws," pp. 544, Cambridge University Press, 1900. In the present paper I shall refer to this volume briefly as the "Treatise".

In various chapters of the Treatise the relations of a system of six screws, of which every pair are reciprocal, have been developed. I have also explained the intimate connexion between the theory of six co-reciprocal screws and the theory of six linear complexes in involution as expounded by German Mathematicians. The object of the present Paper is to give an account of certain further developments of the geometry of six screws, which in some cases are considered as coreciprocal, and in other cases are regarded as quite independent, it being however always understood, unless the contrary is stated, that the six screws are not to be members of a five-system or any lower system.

Let 1, 2, 3, 4, 5, 6 denote six *coreciprocal* screws, of which $p_1, p_2, p_3, p_4, p_5, p_6$ are the pitches. Let sin (12) be the sine of the angle between 1 and 2, and let sin (123) denote the sine of the solid angle contained by 1, 2, 3. Then it has been proved (Treatise, p. 284) that the following formulæ hold good:—

$$\Sigma \frac{1}{p_1} = 0, \tag{i}$$

$$\Sigma \frac{\sin^2(12)}{p_1 p_2} = 0, \tag{ii}$$

$$\Sigma \frac{\sin^2(123)}{p_1 p_2 p_3} = 0, \tag{iii}$$

where the signs of summation include every possible combination of the six screws, so that in the first of these formulæ there are six terms, in the second fifteen, and in the third twenty.

I had mentioned (Treatise, p. 495) that Mr. J. H. Grace, Fellow of St. Peter's College, Cambridge, had kindly pointed out to me that the formula (ii) which I had shown to be true when the summation extended over six coreciprocal screws, would be also true if the summation extended over only five screws, of which each pair were reciprocal. Mr. Grace also mentioned that formula (iii), which I had given for six coreciprocals, would be true if the summation extended to no more than five screws, or even four screws, of which each pair were reciprocal. The significance of these extensions did not at first fully present itself to me, and the Treatise contains accordingly no more than a bare statement to the effect that the extensions were true. But it has since become manifest that these extensions throw a new and interesting light on the geometrical relations of six coreciprocal screws, and I have endeavoured to develop in the early part of this paper the new additions that have thus been made to the theory.

Many investigations in other parts of the theory, on which I have been engaged since the Treatise went to press, have seemed to me of sufficient importance and novelty to be included in this communication. This paper may therefore be regarded as supplementary to the many papers which have already appeared in the publications of the Academy.

§ 1. A proof of the theorem $\Sigma \dfrac{\sin^2(12)}{p_1 p_2} = 0$, whether the summation extend over five screws or over six.

The following proof for the case of six screws is perhaps more direct than that given in Treatise, p. 284. It is founded on the general theorem (see Treatise, p. 287) that if α be any screw whatever, while 1, 2, 3, 4, 5, 6 are six coreciprocals, then

$$\frac{\sin^2(\alpha 1)}{p_1} + \frac{\sin^2(\alpha 2)}{p_2} + \frac{\sin^2(\alpha 3)}{p_3} + \frac{\sin^2(\alpha 4)}{p_4} + \frac{\sin^2(\alpha 5)}{p_5} + \frac{\sin^2(\alpha 6)}{p_6} = 0.$$

Let us take for α successively the six screws 1, 2, 3, 4, 5, 6, and multiply the six equations thus arising by $\dfrac{1}{p_1}, \dfrac{1}{p_2}, \dfrac{1}{p_3}, \dfrac{1}{p_4}, \dfrac{1}{p_5}, \dfrac{1}{p_6}$, respectively, then adding the sums of the products, we have the following equation:—

$$\begin{aligned}
&\qquad\qquad + \frac{\sin^2(12)}{p_1 p_2} + \frac{\sin^2(13)}{p_1 p_3} + \frac{\sin^2(14)}{p_1 p_4} + \frac{\sin^2(15)}{p_1 p_5} + \frac{\sin^2(16)}{p_1 p_6} \\
&+ \frac{\sin^2(21)}{p_2 p_1} \qquad\qquad + \frac{\sin^2(23)}{p_2 p_3} + \frac{\sin^2(24)}{p_2 p_4} + \frac{\sin^2(25)}{p_2 p_5} + \frac{\sin^2(26)}{p_2 p_6} \\
&+ \frac{\sin^2(31)}{p_3 p_1} + \frac{\sin^2(32)}{p_3 p_2} \qquad\qquad + \frac{\sin^2(34)}{p_3 p_4} + \frac{\sin^2(35)}{p_3 p_5} + \frac{\sin^2(36)}{p_3 p_6} \\
&+ \frac{\sin^2(41)}{p_4 p_1} + \frac{\sin^2(42)}{p_4 p_2} + \frac{\sin^2(43)}{p_4 p_3} \qquad\qquad + \frac{\sin^2(45)}{p_4 p_5} + \frac{\sin^2(46)}{p_4 p_6} \\
&+ \frac{\sin^2(51)}{p_5 p_1} + \frac{\sin^2(52)}{p_5 p_2} + \frac{\sin^2(53)}{p_5 p_3} + \frac{\sin^2(54)}{p_5 p_4} \qquad\qquad + \frac{\sin^2(56)}{p_5 p_6} \\
&+ \frac{\sin^2(61)}{p_6 p_1} + \frac{\sin^2(62)}{p_6 p_2} + \frac{\sin^2(63)}{p_6 p_3} + \frac{\sin^2(64)}{p_6 p_4} + \frac{\sin^2(65)}{p_6 p_5} \qquad\qquad = 0.
\end{aligned}$$

Dividing this by 2 we obtain fifteen terms for the six screws, and their sum is zero or $\Sigma \dfrac{\sin^2(12)}{p_1 p_2} = 0$.

The theorem, already mentioned as due to Mr. Grace, expressing that the same formula is true for the ten terms obtained from the combinations in pairs of no more than five screws can be easily shown. The last row in the left-hand side of the equation just written is, of course, zero, like each of the other rows. But it will be noted that the last column has the same terms as the last row. The expression obtained by removing both the last row and the last column must therefore be zero. This takes away every term involving the sixth screw. The remaining twenty terms unite in ten pairs, thus proving that for the ten combinations that can be made of five coreciprocal screws we must have $\Sigma \dfrac{\sin^2(12)}{p_1 p_2} = 0$.

§ 2. If six coreciprocal screws be divided into any two sets, then the sum of all the quantities $\sin^2(12) \div p_1 p_2$, which can be made from the pairs of screws in one of the sets, is equal to the sum of all the quantities of the same form which can be made from the pairs of screws in the other set.

It is obvious that the only two cases which can arise are those in which the sets number two and four, or three and three, respectively. We shall take the former case first.

We write the following equation in which the left-hand side is zero by the theorem of the last article, while the right-hand side is zero by the general principle already employed (Treatise, p. 287):—

$$\begin{aligned}
&+\frac{\sin^2(21)}{p_2 p_1} \\
&+\frac{\sin^2(31)}{p_3 p_1}+\frac{\sin^2(32)}{p_3 p_2} \\
&+\frac{\sin^2(41)}{p_4 p_1}+\frac{\sin^2(42)}{p_4 p_2}+\frac{\sin^2(43)}{p_4 p_3} \\
&+\frac{\sin^2(51)}{p_5 p_1}+\frac{\sin^2(52)}{p_5 p_2}+\frac{\sin^2(53)}{p_5 p_3}+\frac{\sin^2(54)}{p_5 p_4} \\
=&+\frac{\sin^2(51)}{p_5 p_1}+\frac{\sin^2(52)}{p_5 p_2}+\frac{\sin^2(53)}{p_5 p_3}+\frac{\sin^2(54)}{p_5 p_4}+\frac{\sin^2(56)}{p_5 p_6},
\end{aligned}$$

whence, by cancelling four identical terms on each side, we have the first part of the theorem expressed in the formula

$$\begin{aligned}\frac{\sin^2(56)}{p_5 p_6} = &+ \frac{\sin^2(21)}{p_2 p_1} \\ &+ \frac{\sin^2(31)}{p_3 p_1} + \frac{\sin^2(32)}{p_3 p_2} \\ &+ \frac{\sin^2(41)}{p_4 p_1} + \frac{\sin^2(42)}{p_4 p_2} + \frac{\sin^2(43)}{p_4 p_3}.\end{aligned}$$

To prove the second part, we add to the left-hand side of the formula just written, the expression already known to be zero,

$$\frac{\sin^2(41)}{p_4 p_1} + \frac{\sin^2(42)}{p_4 p_2} + \frac{\sin^2(43)}{p_4 p_3} + \frac{\sin^2(45)}{p_4 p_5} + \frac{\sin^2(46)}{p_4 p_6},$$

whence, by cancelling three identical terms on each side, we have

$$\frac{\sin^2(21)}{p_2 p_1} + \frac{\sin^2(31)}{p_3 p_1} + \frac{\sin^2(32)}{p_2 p_3} = \frac{\sin^2(45)}{p_4 p_5} + \frac{\sin^2(46)}{p_4 p_6} + \frac{\sin^2(56)}{p_5 p_6}.$$

Thus the theorem of this article is completely proved.

The fifteen quantities, $\sin^2(12) \div p_1 p_2$, belonging to a coreciprocal system, have thus a series of relations which may perhaps be most concisely described as follows.

Let us assign to each of the quantities, as a characteristic mark, one of the combinations of the six digits 1, 2, 3, 4, 5, 6, taken two together, e.g., one of the quantities will be denoted by (13) and another by (46), &c.

Then the fifteen numbers possess the following properties:—

1. Their sum is zero.

2. The sum of any five numbers, of which each one contains a specified digit, is zero. For example, if 2 be the digit—

$$(21) + (23) + (24) + (25) + (26) = 0.$$

3. The sum of any ten numbers, from which a specified digit is excluded, is zero. For example, if 1 be the digit—

$$(23) + (24) + (25) + (26) + (34) + (35) + (36) + (45) + (46) + (56) = 0.$$

4. If the six digits be divided into two sets, then the sum of the numbers which can be formed by the digits of one set is equal to the sum of the numbers which can be formed by the digits of the other set, e.g.:

$$(12) + (13) + (23) = (45) + (56) + (46);$$

$$(12) = (34) + (35) + (36) + (45) + (46) + (56).$$

The last of these formulæ expresses a geometrical result that may be noted. We learn that $\sin^2(12) \div p_1 p_2$ must remain constant for all pairs of screws which are reciprocal, provided that each of them is also reciprocal to the four screws 3, 4, 5, 6. This means, that 1 and 2 may be any pair of screws which lie on the cylindroid reciprocal to 1 and 2. We thus obtain the following theorem.

If any two reciprocal screws be taken on a cylindroid, the square of the sine of the angle between them, divided by the product of their pitches, is constant.

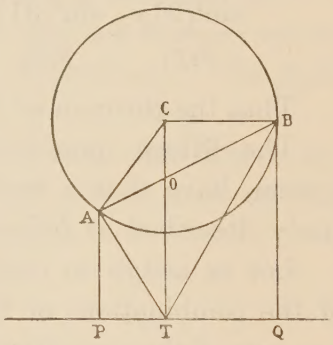

FIG. 1.

A direct proof of this may be obtained at once from the circular representation of the cylindroid (Treatise, p. 45). Let A, B be two reciprocal screws, then their chord AB passes through O, the pole of the axis of pitch, PQ. Then, $\angle ATO = \angle BTO$, and $\sin ATP = \sin BTQ$, whence $AT \cdot BT \sin^2 ATP = AP \cdot BQ$. But $AT \cdot BT = TO^2 + AO \cdot BO$, which is constant for all chords through O; whence $\sin^2 ATP \div AP \cdot BQ$ is constant. But $CO \cdot TO = AO \cdot BO$. Hence the quadrilateral $TACB$ is inscribable in a circle, and consequently ATP is equal to the angle subtended at the circumference by A and B, i.e. to the angle between the screws corresponding to A and B; also, AP and BQ are the pitches of A and B, and the theorem is proved.

§ 3. Let 1, 2, 3, n be any system of coreciprocal screws chosen from an n-system $S(5 > n > 1)$. Let $p_1, \ldots p_n$ be the pitches of these n screws

respectively. Let the symbol Σ mean the summation of every pair that can be formed from the n coreciprocals, then the theorem now enunciated expresses that $\Sigma \sin^2(12) \div p_1 p_2$ has the same value for every one of the different groups of n coreciprocals which can be chosen from S, and further, if S' be the system reciprocal to S, then the constant value of $\Sigma \sin^2(12) \div p_1 p_2$ for a group of $6 - n$ coreciprocals in S', is equal to the constant value of $\Sigma \sin^2(12) \div p_1 p_2$ for a group of n coreciprocals in S.

The proof of this general theorem is an obvious induction from the results already set down. Writing the formula

$$\frac{\sin^2(56)}{p_5 p_6} = \frac{\sin^2(12)}{p_1 p_2} + \frac{\sin^2(13)}{p_1 p_3} + \frac{\sin^2(14)}{p_1 p_4} + \frac{\sin^2(23)}{p_2 p_3} + \frac{\sin^2(24)}{p_2 p_4} + \frac{\sin^2(34)}{p_3 p_4}$$

we have already seen, in the last article, how the theorem is true for the case of $n = 2$. This formula also proves the theorem for the case of $n = 4$. For, we may take as 1, 2, 3, 4 *any* four coreciprocal screws in the system reciprocal to 5 and 6. But the left-hand member of the equation is unchanged by any alteration in the choice of 1, 2, 3, 4, subject to the conditions stated. Hence the right-hand member must remain the same, notwithstanding the change of 1, 2, 3, 4 to any other four coreciprocals. Thus we prove the case of $n = 4$.

For the case of $n = 3$, we have the formula

$$\frac{\sin^2(12)}{p_1 p_2} + \frac{\sin^2(13)}{p_1 p_3} + \frac{\sin^2(23)}{p_2 p_3} = \frac{\sin^2(45)}{p_4 p_5} + \frac{\sin^2(46)}{p_4 p_6} + \frac{\sin^2(56)}{p_5 p_6}.$$

As the right-hand member is unaltered by the change of 1, 2, 3 to any other triad of coreciprocals which belong to the system reciprocal to 4, 5, 6, it follows that this change must produce no effect on the left-hand member also. Thus the theorem is proved for the case of $n = 3$.

In the cases of $n = 5$, and $n = 6$, the theorem is proved from the fact that, in each of these instances, $\Sigma \sin^2(12) \div p_1 p_2$ is zero, as is shown in § 1.

We have thus proved the general theorem enunciated at the beginning of this article.

§ 4. Let $p_1, p_2, p_3, p_4, p_5, p_6$ be the pitches of the six screws of any coreciprocal system. Let $\sin(123)$ be the sine of the solid angle contained by straight lines parallel to the three first screws. Let $\sin(456)$ be similarly related to the remaining three screws. Then we have

$$\frac{\sin^2(123)}{p_1 p_2 p_3} + \frac{\sin^2(456)}{p_4 p_5 p_6} = 0.$$

Other proofs of this theorem will be given later; the proof now given shows the way in which the theorem first presented itself. It depends on the well-known geometrical property, that the volume of the parallelepiped of which any three conjugate semidiameters of a given quadric are conterminous edges, equals the product of the three semi-axes of the quadric.

Let p_1, p_2, p_3 define the pitches of a triad of coreciprocal screws in a three-system, and let OP_1, OP_2, OP_3 be semidiameters of the pitch quadric parallel thereto; then, from Treatise, p. 176, we know that OP_1, OP_2, OP_3 are conjugate diameters of the pitch quadric. We have, accordingly,

$$OP_1 \times OP_2 \times OP_3 \sin(123) = abc.$$

The equation of the pitch quadric, when referred to the three principal screws of the system with pitches p_α, p_β, p_γ, is expressed as follows (Treatise, p. 173):—

$$p_\alpha x^2 + p_\beta y^2 + p_\gamma z^2 + p_\alpha p_\beta p_\gamma = 0.$$

Then we have, from the property of the pitch quadric,

$$p_1 = -\frac{p_\alpha p_\beta p_\gamma}{OP_1^2}; \qquad p_2 = -\frac{p_\alpha p_\beta p_\gamma}{OP_2^2}; \qquad p_3 = -\frac{p_\alpha p_\beta p_\gamma}{OP_3^2};$$

whence

$$\frac{(p_\alpha p_\beta p_\gamma)^3}{p_1 p_2 p_3} \sin^2(123) = -a^2 b^2 c^2.$$

Identifying

$$p_\alpha x^2 + p_\beta y^2 + p_\gamma z^2 + p_\alpha p_\beta p_\gamma = 0,$$

with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

we have
$$a^2 b^2 c^2 = -(p_\alpha p_\beta p_\gamma)^2;$$
whence, finally,
$$\frac{\sin^2(123)}{p_1 p_2 p_3} = \frac{1}{p_\alpha p_\beta p_\gamma}.$$

Let us now consider the other three screws of the coreciprocal system. They must define the same pitch quadric as that already determined. This is obvious, because one set of generators on the quadric belong to one of the three-systems, and the other set of generators belongs to the reciprocal three-system. If a three-system has three principal screws of pitches, p_α, p_β, p_γ, then the reciprocal system will have three principal screws on the same axes, but with pitches $-p_\alpha$, $-p_\beta$, $-p_\gamma$. Hence we obtain
$$\frac{\sin^2(456)}{p_4 p_5 p_6} = -\frac{1}{p_\alpha p_\beta p_\gamma},$$
and thus we prove the formula
$$\frac{\sin^2(123)}{p_1 p_2 p_3} + \frac{\sin^2(456)}{p_4 p_5 p_6} = 0.$$

From this we could have deduced the well-known theorem, due to Klein (Treatise, p. 287), that of the six pitches of a coreciprocal system, three are positive and three negative. For, if four of the pitches, p_1, p_2, p_3, p_4, had the same sign, and p_5, p_6 the opposite sign, it would be impossible to distribute them into two groups of three each, whose products had unlike signs. In a slightly different manner it can be shown to be impossible for the signs to have been alike in five screws, and different in the sixth, or to have been all alike.

We have also another proof of the general formula for six coreciprocals (Treatise, p. 284):
$$\Sigma \frac{\sin^2(123)}{p_1 p_2 p_3} = 0.$$

For, we now see that the twenty terms in this expression can be arranged in ten pairs, such that the sum of each pair is zero.

§ 5. If any four coreciprocals be denoted by 1, 2, 3, 4, and if their pitches be p_1, p_2, p_3, p_4, respectively, and if $\sin(123)$ denotes the sine of the solid angle contained by the screws 1, 2, 3, then the theorem to be proved is that due to Mr. Grace—

$$\Sigma \frac{\sin^2(123)}{p_1 p_2 p_3} = 0.$$

Other proofs of this will be given subsequently. It may, perhaps, be most directly shown as follows—

Let r_1, r_2, r_3 be the lengths of three conjugate semi-diameters of a central quadric. Let r_α be any fourth semi-diameter. Let $\sin(\alpha 12)$ denote the sine of the solid angle formed by 1, 2, and r_α; then, with similar notation for the other solid angles, we have

$$\frac{\sin^2(\alpha 23)}{r_1^2} + \frac{\sin^2(\alpha 31)}{r_2^2} + \frac{\sin^2(\alpha 12)}{r_3^2} = \frac{\sin^2(123)}{r_\alpha^2}.$$

An adaptation of this general formula supplies a direct proof of the present theorem. Let p_α, p_β, p_γ be the pitches of the principal screws of the system formed by the three coreciprocals 1, 2, 3. Then we know that

$$p_1 = -\frac{p_\alpha p_\beta p_\gamma}{r_1^2}. \qquad p_2 = -\frac{p_\alpha p_\beta p_\gamma}{r_2^2}; \qquad p_3 = -\frac{p_\alpha p_\beta p_\gamma}{r_3^2}.$$

But a three-system and its reciprocal have the same pitch quadric. In the original system the pitch of the screw, parallel to r, is $\frac{p_\alpha p_\beta p_\gamma}{r^2}$, while in the reciprocal system the pitch of the screw, parallel to r, has the same magnitude, but the opposite sign. Let r_4 be the semi-diameter parallel to 4, which is a screw of the system reciprocal to 1, 2, 3. We therefore have $p_4 = +\frac{p_\alpha p_\beta p_\gamma}{r_4^2}$, whence the above formula gives

$$p_1 \sin^2(234) + p_2 \sin^2(134) + p_3 \sin^2(124) + p_4 \sin^2(123) = 0,$$

$$\text{or,} \quad \Sigma \frac{\sin^2(123)}{p_1 p_2 p_3} = 0.$$

§ 6. If any five coreciprocals be denoted by 1, 2, 3, 4, 5, respectively, and if their pitches be p_1, p_2, p_3, p_4, p_5, and if $\sin(123)$ denote the sine of the solid angle contained by 1, 2, 3, and similarly for other combinations, then the theorem to be proved is that

$$\Sigma \frac{\sin^2(123)}{p_1 p_2 p_3} = 0.$$

We can deduce this most conveniently from the analogous theorem for four coreciprocals, as proved in the last article. If we omit one of the five screws 1, 2, 3, 4, 5, we have, for the four which remain, the formula given in the last article involving four terms. If now we add together the five sets of four terms, obtained by omitting each one in succession, we obtain 20 terms equal in pairs, and thus we obtain the sum of ten terms equal to zero. These terms correspond to every combination of three which can be made out of the five quantities, and consequently the theorem is proved. This also, as already stated, is due to Mr. Grace.

§ 7. If 1, 2, 3, 4, 5, 6 be any six coreciprocal screws. Then, every formula of the following type is true:

$$\frac{\sin^2(156)}{p_1} + \frac{\sin^2(256)}{p_2} + \frac{\sin^2(356)}{p_3} + \frac{\sin^2(456)}{p_4} = 0.$$

For, write the general theorem $\Sigma \dfrac{\sin^2(123)}{p_1 p_2 p_3} = 0$ in the following form—

$$\begin{aligned}
&\frac{\sin^2(123)}{p_1 p_2 p_3} + \frac{\sin^2(124)}{p_1 p_2 p_4} + \frac{\sin^2(234)}{p_2 p_3 p_4} + \frac{\sin^2(134)}{p_1 p_3 p_4} \\
&+ \left[\frac{\sin^2(235)}{p_2 p_3 p_5} + \frac{\sin^2(614)}{p_1 p_4 p_6}\right] + \left[\frac{\sin^2(345)}{p_3 p_4 p_5} + \frac{\sin^2(612)}{p_1 p_2 p_6}\right] \\
&+ \left[\frac{\sin^2(245)}{p_2 p_4 p_5} + \frac{\sin^2(613)}{p_6 p_1 p_3}\right] + \left[\frac{\sin^2(451)}{p_4 p_5 p_1} + \frac{\sin^2(623)}{p_6 p_2 p_3}\right] \\
&+ \left[\frac{\sin^2(351)}{p_3 p_5 p_1} + \frac{\sin^2(624)}{p_6 p_2 p_4}\right] + \left[\frac{\sin^2(512)}{p_5 p_1 p_2} + \frac{\sin^2(634)}{p_6 p_3 p_4}\right] \\
&+ \frac{\sin^2(156)}{p_1 p_5 p_6} + \frac{\sin^2(256)}{p_2 p_5 p_6} + \frac{\sin^2(356)}{p_3 p_5 p_6} + \frac{\sin^2(456)}{p_4 p_5 p_6} = 0,
\end{aligned}$$

the sum of the first four terms is zero by § 5. Each of the pairs within

the brackets are zero by § 4, and hence the four remaining terms must be also zero. This proves the theorem.

We can also prove this result in the following manner from the general formula (Treatise, p. 287):

$$\frac{\cos^2(a1)}{p_1}+\frac{\cos^2(a2)}{p_2}+\frac{\cos^2(a3)}{p_3}+\frac{\cos^2(a4)}{p_4}+\frac{\cos^2(a5)}{p_5}+\frac{\cos^2(a6)}{p_6}=0.$$

where a is any screw whatever.

We shall take for a a direction parallel to the axis of the cylindroid defined by 5 and 6, so that $\cos(a5)$ and $\cos(a6)$ are both zero. If three unit lines be drawn from a point and parallel to 1, 5, 6, then the volume of the parallelepiped, of which these are conterminous edges, is $\sin(156)$. Remembering that a is normal to the plane of 5 and 6, the projection of a line of unit length on a, along the normal to 5 and 6, is $\cos(1a)$, and this multiplied into $\sin(56)$ must be the volume of the parallelepiped. We thus have

$$\sin(156)=\sin(56)\cos(a1);\qquad \sin(256)=\sin(56)\cos(a2);$$
$$\sin(356)=\sin(56)\cos(a3);\qquad \sin(456)=\sin(56)\cos(a4).$$

Substituting these values for $\cos(a1)$, &c., in the preceding formula, and noticing that the two last terms are zero, we have the required result

$$\frac{\sin^2(156)}{p_1}+\frac{\sin^2(256)}{p_2}+\frac{\sin^2(356)}{p_3}+\frac{\sin^2(456)}{p_4}=0.$$

Since the pitches in a coreciprocal system are generally different, from either 0 or ∞, we may introduce the factor p_5p_6 into the denominators in this equation, and we have

$$\frac{\sin^2(156)}{p_1p_5p_6}+\frac{\sin^2(256)}{p_2p_5p_6}+\frac{\sin^2(356)}{p_3p_5p_6}+\frac{\sin^2(456)}{p_4p_5p_6}=0.$$

§ 8. The properties of the twenty numbers, $\sin^2(123)\div p_1p_2p_3$, which can be derived from a coreciprocal system of six screws, by taking every combination of three.

The numbers in question may be conveniently represented by (123), (134), &c. And the first number may be said to "contain" the digits

1, 2, 3, and the second to "contain" the digits 1, 3, 4. Then we have the following properties:—

1. The sum of the twenty numbers is zero.

This is the original theorem given, as already mentioned, in Treatise, p. 284.

2. There are ten numbers which contain any specified digit from 1 to 6, both inclusive, and the sum of those ten numbers is zero.

For, as the sum of the twenty numbers formed from six screws is zero, and also the sum of the ten numbers formed from five of those screws is zero (§ 6), then the difference of these two sums will be ten numbers, all of which have one digit in common.

3. There are ten numbers which exclude one particular digit, and the sum of those ten is zero.

For this sum must be merely that formed from the remaining five digits which, from § 6, is known to be zero.

4. There are four numbers, each of which includes both of two particular digits, and the sum of those four is zero.

For this is the formula proved in the last article.

5. There are four numbers, each of which excludes both of two particular digits, and the sum of those four is zero (§ 5).

6. There are twelve numbers, each of which includes one and only one of two particular digits, and their sum is zero.

For out of the sum of possible numbers, which amount to twenty, we may select the four which contain both of the digits and the four which contain neither; and as each of these sums is zero, and the sum of the twenty is zero, the remaining twelve which each contain one of the digits must be also zero.

7. There are six numbers which each include one particular digit, and each exclude another particular digit, and their sum is zero.

For there are in all ten numbers which exclude one particular digit. Four of these will not contain the digit that has to be kept in. These four will therefore want two digits, and their sum is zero (§ 5), the six which remain must also have zero for their sum.

8. Any two numbers which include all the six digits will have zero for their sum.

§ 9. A more generalized conception of the twenty numbers.

We have been considering a case of twenty numbers, each of which may be ear-marked as corresponding to one of the combinations of six things taken three together. These numbers possess several properties as here set down. It is worth while inquiring into the general character of a system of numbers which fulfil the eight conditions satisfied by these functions of the six screws of a coreciprocal system. The most general solution of the problem is as follows.

Take any six quantities, A, B, C, D, E, F. There are twenty different ways in which signs can be applied to these six letters, so that three of them shall be positive and three negative. If the sums of the six quantities be formed in each of these twenty ways, the result is a series of numbers which possess the eight properties referred to.

The following is the scheme of signs and the corresponding symbols of the numbers:

(123) =	$+A$	$+B$	$+C$	$-D$	$-E$	$-F$
(246) =	$+A$	$+B$	$-C$	$+D$	$-E$	$-F$
(156) =	$+A$	$+B$	$-C$	$-D$	$+E$	$-F$
(345) =	$+A$	$+B$	$-C$	$-D$	$-E$	$+F$
(256) =	$+A$	$-B$	$+C$	$-D$	$-E$	$+F$
(136) =	$+A$	$-B$	$-C$	$+D$	$-E$	$+F$
(124) =	$+A$	$-B$	$-C$	$-D$	$+E$	$+F$
(236) =	$-A$	$+B$	$-C$	$-D$	$+E$	$+F$
(135) =	$-A$	$-B$	$+C$	$-D$	$+E$	$+F$
(456) =	$-A$	$-B$	$-C$	$+D$	$+E$	$+F$
(145) =	$+A$	$-B$	$+C$	$+D$	$-E$	$-F$
(346) =	$+A$	$-B$	$+C$	$-D$	$+E$	$-F$
(235) =	$+A$	$-B$	$-C$	$+D$	$+E$	$-F$
(234) =	$-A$	$-B$	$+C$	$+D$	$-E$	$+F$
(126) =	$-A$	$-B$	$+C$	$+D$	$+E$	$-F$
(134) =	$-A$	$+B$	$-C$	$+D$	$+E$	$-F$
(245) =	$-A$	$+B$	$+C$	$-D$	$+E$	$-F$
(356) =	$-A$	$+B$	$+C$	$+D$	$-E$	$-F$
(125) =	$-A$	$+B$	$-C$	$+D$	$-E$	$+F$
(146) =	$-A$	$+B$	$+C$	$-D$	$-E$	$+F$

It is easy to verify that this solution will satisfy all the conditions. Obviously, the sum of all is zero, which is the first condition. The second may be illustrated by taking the sum of the ten numbers which contain, for example, the digit 6. They are

$$(246)+(156)+(256)+(136)+(236)+(456)+(346)+(126)+(356)+(146),$$

and their sum is zero. We illustrate the third condition by taking the ten numbers which exclude, for example, 4, and we have

$$(123)+(156)+(256)+(136)+(236)+(135)+(235)+(126)+(356)+(125),$$

which is zero. For the fourth condition we take, for example, the numbers which include both 3 and 6. They are

$$(136)+(236)+(346)+(356),$$

which is zero.

For the fifth condition we take, for example, the numbers which exclude both 1 and 5, and we we have

$$(246)+(236)+(346)+(234),$$

which is also zero.

The sixth condition may be illustrated by the numbers which include either 2 or 3, but not both. They are

$$(246)+(345)+(256)+(136)+(124)+(135)+(356)$$
$$+(125)+(346)+(126)+(134)+(245),$$

which, as expected, is zero.

The seventh condition may be illustrated by taking the sum of the six numbers which contain 4, but exclude 5. They are

$$(246)+(124)+(346)+(234)+(134)+(146),$$

which reduce to zero.

The eighth and last condition is illustrated by

$$(125)+(346)=0.$$

Though there are six letters employed, yet as only the differences are involved, there are only five arbitrary quantities in the choice of a system of twenty numbers which shall fulfil the prescribed conditions. We may, therefore, assume that A, B, C, D, E, F satisfy some further condition.

§ 10. Note on certain screws in a four-system.

We may first observe, that in any four-system, where the screws of reference are coreciprocal, the *eight* screws whose coordinates are represented by

$$\pm\,\theta_1\,,\quad \pm\,\theta_2\,,\quad \pm\,\theta_3\,,\quad \pm\,\theta_4$$

must be crossed by two common transversals. For, as the pitch of every screw in the four-system, is represented by $p_1\theta_1^2 + p_2\theta_2^2 + p_3\theta_3^2 + p_4\theta_4^2$, divided by the quadratic function R, whose value is generally unity (Treatise, p. 36), the eight screws just mentioned must all possess the same pitch. Each of them must therefore intersect the two screws of equal and opposite pitch on the reciprocal cylindroid. We know that (§ 5):

$$p_1 \sin^2(234) + p_2 \sin^2(134) + p_3 \sin^2(124) + p_4 \sin^2(123) = 0.$$

It therefore appears that the eight screws, whose coordinates are

$$\pm \sin(234),\quad \pm \sin(134),\quad \pm \sin(124),\quad \pm \sin(123),$$

must each intersect both of the screws of zero pitch on the reciprocal cylindroid. It has now to be shown that one of these eight screws must lie on the axis of the reciprocal cylindroid, and have a pitch of indeterminate value.

Take three rectangular axes, and from the origin draw four rays parallel to 1, 2, 3, 4. Let the first of these rays have direction cosines a, a', a'', the second b, b', b'', the third c, c', c'', and the fourth d, d', d''. Then the coefficients of x, y, z, in the determinant,

$$\begin{vmatrix} x & y & z & w \\ a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{vmatrix}$$

must be the coordinates of some one of the eight screws just referred to. Let us represent them as

$$+\sin(234),\quad +\sin(134),\quad +\sin(124),\quad +\sin(123).$$

The pitch of the screw under consideration is therefore

$$p_1 \sin^2(234) + p_2 \sin^2(134) + p_3 \sin^2(124) + p_4 \sin^2(123)$$

divided by R, which is generally unity, and which may be expressed by

$$\begin{aligned} R \equiv &+ (a \sin(234) + b \sin(134) + c \sin(124) + d \sin(123))^2 \\ &+ (a' \sin(234) + b' \sin(134) + c' \sin(124) + d' \sin(123))^2 \\ &+ (a'' \sin(234) + b'' \sin(134) + c'' \sin(124) + d'' \sin(123))^2, \end{aligned}$$

but from the definitions of $\sin(234)$, &c., which are the coefficients of x, y, z in the determinant, it is plain that

$$\begin{aligned} a \sin(234) + b \sin(134) + c \sin(124) + d \sin(123) &= 0, \\ a' \sin(234) + b' \sin(134) + c' \sin(124) + d' \sin(123) &= 0; \\ a'' \sin(234) + b'' \sin(134) + c'' \sin(124) + d'' \sin(123) &= 0. \end{aligned}$$

The numerator of the expression for the pitch vanishes by the general theorem of § 5, and as the denominator vanishes also for this particular screw, the pitch assumes the form $0 \div 0$. The only screw in the four-system which can fulfil this condition is the screw which lies on the axis of the reciprocal cylindroid. Thus the theorem is proved.

§ 11. Expressions for the intensities of $n+1$ wrenches which equilibrate.

For convenience in writing the formula, I take the case of $n = 4$. It is, however, obvious that the results are also true for the cases where n is 2 or 3, or 5, or 6.

Let θ_1, θ_2, θ_3, θ_4, θ_5 be the intensities of five wrenches which equilibrate on five screws marked 1, 2, 3, 4, 5, in a four-system, and which are not assumed to be coreciprocal. Let p_1, as usual, denote the pitch of the screw 1, and let ϖ_{12} denote, as usual, the virtual coefficient of 1 and 2, with similar notation for the other screws. Then a twist of small amplitude ϵ, about the screw 1, will do against the wrenches on 1, 2, 3, 4, 5 a quantity of work represented by

$$\epsilon\,(p_1\theta_1 + \varpi_{12}\theta_2 + \varpi_{13}\theta_3 + \varpi_{14}\theta_4 + \varpi_{15}\theta_5).$$

But as $\theta_1, \ldots \theta_5$ equilibrate, this expression must be zero. A like

result is obtained by imparting a small twist about each of the other screws, and consequently we have the following system:

$$\begin{aligned}
p_1\,\theta_1 + \varpi_{12}\theta_2 + \varpi_{13}\theta_3 + \varpi_{14}\theta_4 + \varpi_{15}\theta_5 &= 0,\\
\varpi_{21}\theta_1 + p_2\,\theta_2 + \varpi_{23}\theta_3 + \varpi_{24}\theta_4 + \varpi_{25}\theta_5 &= 0,\\
\varpi_{31}\theta_1 + \varpi_{32}\theta_2 + p_3\,\theta_3 + \varpi_{34}\theta_4 + \varpi_{35}\theta_5 &= 0, \qquad \text{i.}\\
\varpi_{41}\theta_1 + \varpi_{42}\theta_2 + \varpi_{43}\theta_3 + p_4\,\theta_4 + \varpi_{45}\theta_5 &= 0,\\
\varpi_{51}\theta_1 + \varpi_{52}\theta_2 + \varpi_{53}\theta_3 + \varpi_{54}\theta_4 + p_5\,\theta_5 &= 0.
\end{aligned}$$

By the elimination of $\theta_1 \ldots \theta_6$, we obtain the following determinantal equation which must be satisfied by any five screws belonging to a four-system,

$$\begin{vmatrix}
p_1 & \varpi_{12} & \varpi_{13} & \varpi_{14} & \varpi_{15}\\
\varpi_{21} & p_2 & \varpi_{23} & \varpi_{24} & \varpi_{25}\\
\varpi_{31} & \varpi_{32} & p_3 & \varpi_{34} & \varpi_{35}\\
\varpi_{41} & \varpi_{42} & \varpi_{43} & p_4 & \varpi_{45}\\
\varpi_{51} & \varpi_{52} & \varpi_{53} & \varpi_{54} & p_5
\end{vmatrix} = 0.$$

As this determinant is zero, we must have, from a property of determinants due, I believe, to Spottiswoode (*Comptes Rendus*, vol. LXVI., pp. 97–103: 1868), the values of θ_1^2, θ_2^2, . . respectively proportional to the corresponding principal minors of this determinant; or, if H be a common factor:

$$H\theta_1^2 = \begin{vmatrix}
p_2 & \varpi_{23} & \varpi_{24} & \varpi_{25}\\
\varpi_{32} & p_3 & \varpi_{34} & \varpi_{35}\\
\varpi_{42} & \varpi_{43} & p_4 & \varpi_{45}\\
\varpi_{52} & \varpi_{53} & \varpi_{54} & p_5
\end{vmatrix} \qquad H\theta_2^2 = \begin{vmatrix}
p_1 & \varpi_{13} & \varpi_{14} & \varpi_{15}\\
\varpi_{31} & p_3 & \varpi_{34} & \varpi_{35}\\
\varpi_{41} & \varpi_{43} & p_4 & \varpi_{45}\\
\varpi_{51} & \varpi_{53} & \varpi_{54} & p_5
\end{vmatrix}$$

with similar values for θ_3^2, θ_4^2, and θ_5^2.

We can write a more general property of a four-system by the following considerations. It is necessary for the equilibrium of the wrenches, that

the five forces $\theta_1 \ldots \theta_5$ should, if applied at a point, equilibrate each other, and accordingly

$$\left.\begin{aligned}
\theta_1 + \cos(12)\theta_2 + \cos(13)\theta_3 + \cos(14)\theta_4 + \cos(15)\theta_5 &= 0,\\
\cos(21)\theta_1 + \theta_2 + \cos(23)\theta_3 + \cos(24)\theta_4 + \cos(25)\theta_5 &= 0,\\
\cos(31)\theta_1 + \cos(32)\theta_2 + \theta_3 + \cos(34)\theta_4 + \cos(35)\theta_5 &= 0,\\
\cos(41)\theta_1 + \cos(42)\theta_2 + \cos(43)\theta_3 + \theta_4 + \cos(45)\theta_5 &= 0,\\
\cos(51)\theta_1 + \cos(52)\theta_2 + \cos(53)\theta_3 + \cos(54)\theta_4 + \theta_5 &= 0.
\end{aligned}\right\} \text{ ii.}$$

We thus have, in i. and ii., ten equations from any five of which $\theta_1 \ldots \theta_5$ may be eliminated. We may include all the results in the following determinant, which is obtained by multiplying the five equations of ii. by x_1, x_2, x_3, x_4, x_5, respectively, adding them to the equations of i. and then eliminating $\theta_1, \ldots \theta_5$. The result is

$$\begin{vmatrix}
p_1 + x_1 & \varpi_{12} + x_1\cos(12) & \varpi_{13} + x_1\cos(13) & \varpi_{14} + x_1\cos(14) & \varpi_{15} + x_1\cos(15)\\
\varpi_{12} + x_2\cos(21) & p_2 + x_2 & \varpi_{23} + x_2\cos(23) & \varpi_{24} + x_2\cos(24) & \varpi_{25} + x_2\cos(25)\\
\varpi_{13} + x_3\cos(31) & \varpi_{32} + x_3\cos(32) & p_3 + x_3 & \varpi_{34} + x_3\cos(34) & \varpi_{35} + x_3\cos(35)\\
\varpi_{14} + x_4\cos(41) & \varpi_{42} + x_4\cos(42) & \varpi_{43} + x_4\cos(43) & p_4 + x_4 & \varpi_{45} + x_4\cos(45)\\
\varpi_{15} + x_5\cos(51), & \varpi_{52} + x_5\cos(52), & \varpi_{53} + x_5\cos(53), & \varpi_{54} + x_5\cos(54), & p_5 + x_5
\end{vmatrix}$$

If this determinant be expanded, then every coefficient of terms containing x_1, x_2, x_3, x_4, x_5 must vanish identically.

In the case of four equilibrating wrenches on four screws belonging to a three-system,

$$\begin{aligned}
\theta_1 + \cos(12)\theta_2 + \cos(13)\theta_3 + \cos(14)\theta_4 &= 0,\\
\cos(21)\theta_1 + \theta_2 + \cos(23)\theta_3 + \cos(24)\theta_4 &= 0,\\
\cos(31)\theta_1 + \cos(32)\theta_2 + \theta_3 + \cos(34)\theta_4 &= 0,\\
\cos(41)\theta_1 + \cos(42)\theta_2 + \cos(43)\theta_3 + \theta_4 &= 0.
\end{aligned}$$

We then have, by the property of determinants already employed,

$$H\theta_1^2 = \begin{vmatrix} 1 & \cos(23) & \cos(24) \\ \cos(32) & 1 & \cos(34) \\ \cos(42) & \cos(43) & 1 \end{vmatrix}, \quad H\theta_2^2 = \begin{vmatrix} 1 & \cos(13) & \cos(14) \\ \cos(13) & 1 & \cos(34) \\ \cos(14) & \cos(43) & 1 \end{vmatrix}$$

Let $\alpha_1, \alpha_2, \alpha_3$; $\beta_1, \beta_2, \beta_3$; $\gamma_1, \gamma_2, \gamma_3$ be the direction cosines of 2, 3, 4. Then, as is well known,

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}^2 = \begin{vmatrix} 1 & \cos(23) & \cos(24) \\ \cos(32) & 1 & \cos(34) \\ \cos(42) & \cos(43) & 1 \end{vmatrix}$$

but the determinant on the left is but an expression for $\sin(234)$. Hence we have the important results

$$H\theta_1^2 = \sin^2(234); \quad H\theta_2^2 = \sin^2(134); \quad H\theta_3^2 = \sin^2(124); \quad H\theta_4^2 = \sin^2(123);$$

and thus we obtain the theorem which was first communicated to me by Professor C. J. Joly, and which may be thus stated.

If wrenches on four screws belonging to a three-system are in equilibrium, then the intensity of each wrench must be proportional to the sine of the solid angle contained by the three non-corresponding screws.

From this theorem we obtain convenient expressions for the coordinates of any screw λ in a three-system, when referred to any triad of coreciprocal screws of reference,

$$\lambda_1 = \pm \frac{\sin(23\lambda)}{\sin(123)}, \quad \lambda_2 = \pm \frac{\sin(13\lambda)}{\sin(123)}, \quad \lambda_3 = \pm \frac{\sin(12\lambda)}{\sin(123)};$$

and, for the pitch of λ,

$$\sin^2(123)p_\lambda = \sin^2(23\lambda)p_1 + \sin^2(13\lambda)p_2 + \sin^2(12\lambda)p_3.$$

It is interesting to note how the fundamental theorem of § 5 can be also derived from this result. In the equation just given, λ is understood to belong to the system defined by the coreciprocals 1, 2, 3. But in the

system reciprocal to 1, 2, 3 there will be a screw parallel to λ, but with a pitch equal and opposite thereto. If we call this screw 4, then $p_4 = -p_\lambda$, and we have

$$0 = \sin^2(234)p_1 + \sin^2(134)p_2 + \sin^2(124)p_3 + \sin^2(123)p_4;$$

as 1, 2, 3, 4 are any set of coreciprocals, this proof is general.

Returning to the three-system, we have in the expression for the pitch an easy way of arriving at the pitch quadric (Treatise, p. 172). If γ be the diagonal of a parallelepiped, and x, y, z the lengths of its conterminous sides, then

$$\frac{x}{r} = \frac{\sin(ryz)}{\sin(xyz)}; \qquad \frac{y}{r} = \frac{\sin(rzx)}{\sin(xyz)}; \qquad \frac{z}{r} = \frac{\sin(rxy)}{\sin(xyz)}.$$

If, therefore, the radius vector r be parallel to λ, and if the components x, y, z be parallel to 1, 2, 3, we have

$$r^2 p_\lambda = x^2 p_1 + y^2 p_2 + z^2 p_3.$$

Regarding x, y, z as the coordinates of a point on the quadric surface represented by

$$p_1 x^2 + p_2 y^2 + p_3 z^2 = H,$$

we obtain the fundamental property of the pitch quadric

$$p_\lambda = \frac{H}{r^2},$$

and this is, perhaps, the most direct method of obtaining it. The quantity H can be determined as follows.

Let p_α, p_β, p_γ be the principal pitches of the system, and let ρ be the radius vector, so that

$$\rho^2 = x^2 + y^2 + z^2 + 2xy\cos(12) + 2xz\cos(13) + 2yz\cos(23);$$

then, for a maximum or minimum value of ρ, we have, by introducing the common factor λ,

$$\begin{aligned} x + y\cos(12) + z\cos(13) &= \lambda p_1 x, \\ x\cos(21) + y + z\cos(32) &= \lambda p_2 y, \\ x\cos(31) + y\cos(32) + z &= \lambda p_3 z; \end{aligned}$$

whence, by elimination of x, y, z, we have a cubic for λ, and if λ_1, λ_2, λ_3 be the three roots, then, from the absolute term of this equation,

$$\lambda_1\lambda_2\lambda_3 = \frac{\sin^2(123)}{p_1p_2p_3}.$$

If ρ_1^2, ρ_2^2, ρ_3^2 be the squares of the greatest and least semi-axes of the quadric, then we have

$$\lambda_1 H = \rho_1^2, \quad \lambda_2 H = \rho_2^2, \quad \lambda_3 H = \rho_3^2;$$

whence $\lambda_1\lambda_2\lambda_3 H^3 = \rho_1^2\rho_2^2\rho_3^2$. But $\rho_1^2 = -p_\beta p_\gamma$; $\rho_2^2 = -p_\alpha p_\gamma$; $\rho_3^2 = -p_\alpha p_\beta$; whence $H = -\dfrac{p_1p_2p_3}{\sin^2(123)}$, and we see that the equation of the pitch quadric referred to three axes through the centre of the system, and which are parallel to a set of coreciprocal screws, denoted by 1, 2, 3, is

$$p_1x^2 + p_2y^2 + p_3z^2 + \frac{p_1p_2p_3}{\sin^2(123)} = 0.$$

We have also the following theorem, inasmuch as the same pitch quadric is obtained by taking any three of a coreciprocal system of six screws, as is found by taking the three remaining screws (Treatise, p. 174):

Let 1, 2, 3, 4, 5, 6 be six coreciprocals. Draw, through any point O, three axes parallel to 1, 2, 3, and form the quadric, whose equation is

$$p_1x^2 + p_2y^2 + p_3z^2 + \frac{p_1p_2p_3}{\sin^2(123)} = 0. \qquad \text{i.}$$

Through the same point O, draw three other axes parallel to 4, 5, 6, and from the quadric whose equation is

$$p_4x^2 + p_5y^2 + p_6z^2 + \frac{p_4p_5p_6}{\sin^2(456)} = 0; \qquad \text{ii.}$$

then these two quadrics are identical.

Let a point whose coordinates are x, y, z, with respect to the three

first axes, have coordinates x', y', z', with respect to the other three axes. It is easy to see that

$$\begin{aligned}
\sin(123)x &= \sin(423)x' + \sin(523)y' + \sin(623)z', \\
\sin(123)y &= \sin(413)x' + \sin(513)y' + \sin(613)z', \\
\sin(123)z &= \sin(412)x' + \sin(512)y' + \sin(612)z'.
\end{aligned}$$

This transformation must change i. into ii. Observing that the two absolute terms differ only in sign, we have another proof that

$$p_1 \sin^2(234) + p_2 \sin^2(134) + p_3 \sin^2(124) + p_4 \sin^2(123) = 0;$$

and we also obtain a number of formulæ, of which the following is a type:

$$p_1 \sin(423)\sin(523) + p_2 \sin(413)\sin(513) + p_3 \sin(412)\sin(512) = 0.$$

§ 12. On screws of stationary pitch.

It is necessary to discuss here some relations of screws whose pitch is of stationary value in comparison with the pitches of other screws in the vicinity and belonging to the same system. The subject is examined in Treatise, p. 221. We are now to give some further developments. We shall first prove that each screw of stationary pitch in a three-system, must be a principal screw of the system, and conversely. To do this we may make use of the plane representation (Treatise, p. 197).

Let P be a point which corresponds in the plane representation to a screw of the three-system. If P is to be a screw of stationary pitch, in the three-system, it must, *a fortiori*, be a stationary screw on every cylindroid in the three-system which passes through P. Each ray of the pencil which can be drawn through P, corresponds to one of the cylindroids of the system, so that if P be a screw of stationary pitch in the three-system, then every one of the cylindroids through P must have P as one of its principal screws.

Draw the pitch-conic through P, and draw through P a tangent to that pitch-conic. Then P will be one of the principal screws of the cylindroid corresponding to that tangent, and unless the line through P is a tangent to the pitch-conic through that point, then P is not a principal screw of the cylindroid. If, therefore, P is to be a stationary

screw of the three-system, every straight line through P must intersect the pitch-conic through P in two coincident points. This condition can only be fulfilled when the pitch-conic breaks up into two straight lines which have P as their point of intersection.

Thus P must be one of the vertices of the common conjugate triangle of the pitch conics, i.e., P must be one of the principal screws of the three-system (Treatise, p. 208).

Conversely, each principal screw of the three-system will be the stationary screw on every cylindroid of the system drawn through it, and must therefore be a stationary screw of the three-system itself.

We may prove the same theorem geometrically as follows, without the aid of the plane representation.

Let θ be the stationary screw, and let α and β be any other two screws of the three-system, chosen arbitrarily, but avoiding the case where α, β, and θ are co-cylindroidal.

Draw the cylindroid $(\theta\alpha)$, then as θ is a stationary screw on the three-system, it is, *a fortiori*, a screw of stationary pitch on $(\theta\alpha)$, and consequently θ must be one of the two principal screws on the cylindroid $(\theta\alpha)$. Let λ be the other principal screw on $(\theta\alpha)$, then λ intersects θ at right angles.

In like manner, θ must be a principal screw on the cylindroid $(\theta\beta)$, and if ω be the other principal screw, then ω must intersect θ at right angles. It follows that θ must lie on the axis of the cylindroid (λ, ω). Let ϕ, ψ be the two principal screws of this cylindroid, then they are at right angles and they intersect, and θ passes through the point of intersection and is perpendicular to the plane of ϕ and ψ. Thus θ, ϕ, ψ intersect at right angles, and they all belong to the three-system. They are, therefore, its three principal screws, and the theorem is proved.

§ 13. Any straight line which intersects at right angles $n-1$ screws of an n-system, must be the seat of a screw belonging to the $(6-n)$ system reciprocal to the given n-system.

This theorem may be regarded as a lemma preliminary to the following article. The proof is as follows. Let T be a straight line

which intersects at right angles $n-1$ screws $S_1, \ldots S_{n-1}$ belonging to an n-system. Take any other screw S_n which is independent of $S_n \ldots S_{n-1}$, but like them belongs to the n-system. Let θ be a screw located on T, and having its pitch so chosen that it is reciprocal to S_n. Then θ is reciprocal to each of the screws $S_1, \ldots S_{n-1}$, because it intersects each of them at right angles. It is further reciprocal to S_n by construction. It is, therefore, reciprocal to $S_1, \ldots S_n$, i.e., it must belong to the reciprocal system. But θ lies on T, and thus the required theorem has been proved.

As an example of this theorem, it may be mentioned that any ray which intersects at right angles three screws belonging to a four-system, must lie on one of the generators of the reciprocal cylindroid.

§ 14. If in an n-system the screw θ have a stationary pitch $+p_\theta$ (the case of infinite pitches being excluded), then a screw with a pitch $-p_\theta$ on the same straight line as θ, must belong to the reciprocal $(6-n)$-system.

This is, perhaps, the most fundamental principle in the theory of the screws of stationary pitch. We shall prove it geometrically as follows.

Let $(n-1)$ screws $S_1, \ldots S_{n-1}$ be chosen from the n-system, these screws being all independent, by which we mean that, if taken in conjunction with the screw of stationary pitch θ, they are not all contained in an $(n-1)$-system.

Draw the cylindroids $(\theta S_1), \ldots (\theta S_{n-1})$, then θ must, as already explained, be a principal screw on each of these cylindroids. If T_1 be the second principal screw on the cylindroid (θS_1), then θ must intersect T_1 and at right angles. In like manner, θ intersects at right angles the similar screws $T_2, \ldots T_{n-1}$ formed from the cylindroids $(\theta S_2), \ldots (\theta S_{n-1})$. Hence, by the Lemma in the last article, θ must be the seat of a screw belonging to the reciprocal system. But two collinear screws belonging to reciprocal screw-systems must have equal and opposite pitches. Hence we see that a screw on the same straight line as θ, but with a pitch equal and opposite to that of θ, must be a screw of the reciprocal system. This is the theorem required.

Let θ be a screw of stationary pitch. Then, as shown in Treatise, p. 278, and giving R its usual meaning, the screw with coordinates

$$\theta_1 - \frac{p_\theta}{2p_1}\frac{dR}{d\theta_1}, \quad . \quad . \quad . \qquad \theta_6 - \frac{p_\theta}{2p_6}\frac{dR}{d\theta_6}$$

lies on the same axis as θ and has a pitch $-p_\theta$. It must therefore belong to the reciprocal system. If $\alpha_1 \ldots \alpha_6$ be any screw of the original system, then

$$\Sigma\, p_1\alpha_1\left(\theta_1 - \frac{p_\theta}{2p_1}\frac{dR}{d\theta_1}\right) = 0,$$

which reduces to the formula otherwise demonstrated in Treatise, p. 222,

$$\varpi_{\alpha\theta} - p_\alpha \cos(\alpha\theta) = 0.$$

This is the general relation between a stationary screw θ and every other screw α of the n-system.

In the case of freedom of the second order, the theorem of this article shows that, if θ be a screw of stationary pitch p_θ, then a screw collinear with θ and of pitch $-p_\theta$ must be reciprocal to the cylindroid. This requires, in general, that the other screw on the cylindroid, intersected by θ, must be cut at right angles. We have thus another proof that the only stationary screws on a cylindroid are the principal screws.

In the case of freedom of the third order, we see that if θ be a screw of stationary pitch p_θ, then a screw collinear with θ and of pitch $-p_\theta$, is on the reciprocal system, but such cases only occur when θ is one of the three principal screws of the system.

We may state the theorem proved in this article in the following manner, which is perhaps more convenient.

If a screw of pitch $+p$ on a straight line L be a screw of stationary pitch with regard to either of two coreciprocal screw-systems, then a screw of pitch $-p$, on the same line L, will be a screw of stationary pitch on the other coreciprocal system.

This shows that a stationary screw in a four-system will become, if its sign be changed, a stationary screw on the reciprocal cylindroid. It also appears that the only stationary screws in a four-system, will be the two principal screws of the reciprocal cylindroid with their signs changed.

It is worth while to verify this by actual calculation. Let p_1, p_2, p_3, p_4 be the pitches of any four coreciprocal screws which we may take as screws of reference. Let $\theta_1, \theta_2, \theta_3, \theta_4$ be the coordinates of any other screw in the four-system. Then, for the pitch p_θ we have the following equation, which has been rendered homogeneous,

$$\begin{aligned} p_\theta(\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 + 2\theta_1\theta_2\cos(12) + 2\theta_1\theta_3\cos(13) + 2\theta_1\theta_4\cos(14) \\ + 2\theta^2\theta_3\cos(23) + 2\theta_3\theta_4\cos(34) + 2\theta_2\theta_4\cos(24)) \\ - p_1\theta_1^2 - p_2\theta_2^2 - p_3\theta_3^2 - p_4\theta_4^2 = 0. \end{aligned}$$

With the equation in this form, we may treat $\theta_1, \theta_2, \theta_3, \theta_4$ as independent variables. If, then, θ is a screw of stationary pitch, we must be able to vary θ_1 slightly while preserving $\theta_2, \theta_3, \theta_4$ unaltered, and the change must be without effect on p_θ. Differentiating, we have

$$p_\theta\{\theta_1 + \theta_2\cos(12) + \theta_3\cos(13) + \theta_4\cos(14)\} - p_1\theta_1 = 0.$$

There will be three other similar equations from which we eliminate $\theta_1, \theta_2, \theta_3, \theta_4$, and we have

$$\begin{vmatrix} p_\theta - p_1 & \cos(12)p_\theta & \cos(13)p_\theta & \cos(14)p_\theta \\ \cos(21)p_\theta & p_\theta - p_2 & \cos(23)p_\theta & \cos(24)p_\theta \\ \cos(31)p_\theta & \cos(32)p_\theta & p_\theta - p_3 & \cos(34)p_\theta \\ \cos(41)p_\theta & \cos(42)p_\theta & \cos(43)p_\theta & p_\theta - p_4 \end{vmatrix} = 0.$$

This is the equation for p_θ, and we proceed to develop it. The coefficient of p_θ^4 is the determinant

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) & \cos(14) \\ \cos(21) & 1 & \cos(23) & \cos(24) \\ \cos(31) & \cos(32) & 1 & \cos(34) \\ \cos(41) & \cos(42) & \cos(43) & 1 \end{vmatrix}$$

but this is well known to be zero (Treatise, pp. 283, 495); as to the coefficient of p_θ^3, it is

$$-p_1\sin^2(234) - p_2\sin^2(134) - p_3\sin^2(124) - p_4\sin^2(123),$$

but this is also zero (§ 5).

The coefficient of p_θ^2 is easily seen to be

$$p_1 p_2 p_3 p_4 \left(\frac{\sin^2(12)}{p_1 p_2} + \frac{\sin^2(13)}{p_1 p_3} + \frac{\sin^2(14)}{p_1 p_4} + \frac{\sin^2(24)}{p_2 p_4} + \frac{\sin^2(34)}{p_3 p_4} + \frac{\sin^2(23)}{p_2 p_3} \right),$$

but this admits of a remarkable reduction, for if 5 and 6 be two screws on the reciprocal cylindroid, then the six terms in the bracket reduce to (§ 3).

$$\frac{\sin^2(56)}{p_5 p_6}.$$

The coefficient of p_θ is

$$-p_1 p_2 p_3 - p_1 p_2 p_4 - p_1 p_3 p_4 - p_2 p_3 p_4 ;$$

but, from the fundamental theorem that the sum of the reciprocals of the pitches of six coreciprocal screws is equal to zero, we may replace this by

$$p_1 p_2 p_3 p_4 \left(\frac{1}{p_5} + \frac{1}{p_6} \right),$$

and the absolute term is $p_1 p_2 p_3 p_4$. If we reject this factor we have, as the equation for p_θ

$$p_\theta^2 \frac{\sin^2(56)}{p_5 p_6} + p_\theta \left(\frac{1}{p_5} + \frac{1}{p_6} \right) + 1 = 0.$$

There are thus only two screws of stationary pitch in the four-system. If we had commenced directly to calculate the screw ϕ of stationary pitch on the reciprocal cylindroid, we should have for pitch p_ϕ the equation

$$\begin{vmatrix} p_\phi - p_5 & \cos(56) p_\phi \\ \cos(56) p_\phi, & p_\phi - p_6 \end{vmatrix} = 0.$$

or

$$p_\phi^2 \frac{\sin^2(56)}{p_5 p_6} - p_\phi \left(\frac{1}{p_5} + \frac{1}{p_6} \right) + 1 = 0.$$

But this equation has roots equal in magnitude and opposite in sign to those of the former equation. This is the required verification.

§ 15. An extension of the theorem of § 5.

The formula

$$p_1 \sin^2(234) + p_2 \sin^2(134) + p_3 \sin^2(124) + p_4 \sin^2(123) = 0$$

is of course only true for four coreciprocals. We shall now obtain the more general formula, in the case where 1, 2, 3, 4 are not coreciprocal. Let θ be any screw of the four-system, and let a wrench of intensity θ'' act upon θ, and let the components of this wrench, on the four screws of reference, be θ''_1, θ''_2, θ''_3, θ''_4. Let there be a twist of amplitude θ' about the same screw θ, and let $\rho = \theta'' \div \theta'$. Then, from the fundamental principle of the identity of the laws of composition of twists and wrenches, it follows that as the components of the twist have amplitudes θ'_1, θ'_2, θ'_3, θ'_4, the components of the wrench on the same screw will be $\rho\,\theta'_1$, $\rho\,\theta'_2$, $\rho\,\theta'_3$, $\rho\,\theta'_4$. The work done by the wrench θ'', against the twist θ', is $2p_\theta\rho\theta'^2_1$, but this will be equal to the sum of the works done by each component twist against each component wrench. We therefore have

$$p_\theta\theta'^2 = \varpi_{11}\theta'^2_1 + \varpi'^2_{22} + \ldots + 2\varpi_{12}\theta'_1\theta'_2 + \ldots$$

but from the fundamental principle of the composition of twists

$$\theta'^2 = \theta'^2_1 + \theta'^2_2 + \ldots + 2\cos(12)\theta'_1\theta'_2 + \ldots$$

Hence we have the following equation

$$\begin{aligned} p_\theta(\theta'^2_1 + \theta'^2_2 + \theta'^2_3 + \theta'^2_4 + 2\cos(12)\theta'_1\theta'_2 + 2\cos(13)\theta'_1\theta'_3 + 2\cos(14)\theta'_1\theta'_4 \\ + 2\cos(23)\theta'_2\theta'_3 + 2\cos(24)\theta'_2\theta'_4 + 2\cos(34)\theta'_3\theta'_4) \\ = \varpi_{11}\theta'^2_1 + \varpi_{22}\theta'^2_2 + \varpi_{33}\theta'^2_3 + \varpi_{44}\theta'^2_4 + 2\varpi_{12}\theta'_1\theta'_2 + 2\varpi_{13}\theta'_1\theta_3 \\ + 2\varpi_{14}\theta'_1\theta'_4 + 2\varpi_{23}\theta'_2\theta'_3 + 2\varpi_{34}\theta'_3\theta'_4 + 2\varpi_{24}\theta'_2\theta'_4, \end{aligned}$$

whence, as before, we shall have for the screws of stationary pitch the determinantal equation

$$\begin{vmatrix} \varpi_{11} - p_\theta & \varpi_{12} - \cos(12)p_\theta & \varpi_{13} - \cos(13)p_\theta & \varpi_{14} - \cos(14)p_\theta \\ \varpi_{21} - \cos(21)p_\theta & \varpi_{22} - p_\theta & \varpi_{23} - \cos(23)p_\theta & \varpi_{24} - \cos(24)p_\theta \\ \varpi_{31} - \cos(31)p_\theta & \varpi_{32} - \cos(32)p_\theta & \varpi_{33} - p_\theta & \varpi_{34} - \cos(34)p_\theta \\ \varpi_{41} - \cos(41)p_\theta & \varpi_{42} - \cos(42)p_\theta & \varpi_{43} - \cos(43)p_\theta & \varpi_{44} - p_\theta \end{vmatrix} = 0.$$

As we have already seen, the coefficients of p_θ^4 and of p_θ^3 must both vanish. The latter is composed of four determinants, of which the following is a type

$$\begin{vmatrix} \varpi_{11} & \cos(12) & \cos(13) & \cos(14) \\ \varpi_{21} & 1 & \cos(23) & \cos(24) \\ \varpi_{31} & \cos(32) & 1 & \cos(34) \\ \varpi_{41} & \cos(42) & \cos(43) & 1 \end{vmatrix}.$$

This will give the coefficient of ϖ_{11}, and a part of that of ϖ_{21}, the rest of the latter comes from

$$\begin{vmatrix} 1 & \varpi_{12} & \cos(13) & \cos(14) \\ \cos(21) & \varpi_{22} & \cos(23) & \cos(24) \\ \cos(31) & \varpi_{32} & 1 & \cos(34) \\ \cos(41) & \varpi_{42} & \cos(43) & 1 \end{vmatrix}.$$

Thus the coefficients of ϖ_{11} and ϖ_{12} are, respectively,

$$\begin{vmatrix} 1 & \cos(23) & \cos(24) \\ \cos(32) & 1 & \cos(34) \\ \cos(42) & \cos(43) & 1 \end{vmatrix}, \text{ and } -2\begin{vmatrix} \cos(21) & \cos(23) & \cos(24) \\ \cos(31) & 1 & \cos(34) \\ \cos(41) & \cos(43) & 1 \end{vmatrix}.$$

But these will admit of a much more concise expression. Let θ_1, θ_2, θ_3, θ_1 be four quantities determined by the equations

$$\begin{aligned} \theta_1 + \cos(12)\theta_2 + \cos(13)\theta_3 + \cos(14)\theta_4 &= 0, \\ \cos(21)\theta_1 + \theta_2 + \cos(23)\theta_3 + \cos(24)\theta_4 &= 0, \\ \cos(31)\theta_1 + \cos(32)\theta_2 + \theta_3 + \cos(34)\theta_4 &= 0, \\ \cos(41)\theta_1 + \cos(42)\theta_2 + \cos(43)\theta_3 + \theta_4 &= 0. \end{aligned}$$

Then, from Spottiswoode's theorem (§ 11), we can represent the two determinants just written by $h\theta_1^2$ and $-h\theta_1\theta_2$ respectively, with similar

values for the other terms, and where h is a common factor. We thus have the following formula

$$\varpi_{11}\theta_1^2 + \varpi_{22}\theta_2^2 + \varpi_{33}\theta_3^2 + \varpi_{44}\theta_4^2 + 2\varpi_{12}\theta_1\theta_2 + 2\varpi_{13}\theta_1\theta_3 + 2\varpi_{14}\theta_1\theta_4 + 2\varpi_{23}\theta_2\theta_3 + 2\varpi_{24}\theta_2\theta_4 + 2\varpi_{34}\theta_3\theta_4 = 0.$$

If the screws be reciprocal, this reduces to § 5.

It may be here mentioned that wrenches of intensities θ_1, θ_2, θ_3, θ_4 respectively, on the four screws, will have a couple for their resultant.

§ 16. Investigations in a three-system.

If we search for the three screws of stationary pitch p_θ in the three-system defined by three coreciprocal screws 1, 2, 3, we obtain, in the manner shown in § 14, the equation

$$\begin{vmatrix} p_\theta - p_1 & p_\theta\cos(12) & p_\theta\cos(13) \\ p_\theta\cos(21) & p_\theta - p_2 & p_\theta\cos(23) \\ p_\theta\cos(31) & p_\theta\cos(32) & p_\theta - p_3 \end{vmatrix} = 0.$$

Developing this, we find

$$p_\theta^3 \begin{vmatrix} 1 & \cos(12) & \cos(13) \\ \cos(12) & 1 & \cos(23) \\ \cos(13) & \cos(23) & 1 \end{vmatrix} - p_\theta^2\{p_1\sin^2(23) + p_2\sin^2(13) + p_3\sin^2(12)\}$$
$$+ p_\theta(p_1p_2 + p_1p_3 + p_2p_3) - p_1p_2p_3 = 0,$$

which may be written

$$p_\theta^3\frac{\sin^2(123)}{p_1p_2p_3} - p_\theta^2\left(\frac{\sin^2(23)}{p_2p_3} + \frac{\sin^2(13)}{p_1p_3} + \frac{\sin^2(12)}{p_1p_2}\right)$$
$$+ p_\theta\left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}\right) - 1 = 0.$$

We can draw at once a remarkable consequence from this equation. We know that the values of p_θ are the three principal pitches of the system, and we must of course be led to the same three screws what-

ever be the triad of coreciprocals from which we started. Hence we infer that, for any three coreciprocal screws in a three-system, the three following quantities must be constant:

$$1. \quad \frac{\sin^2(123)}{p_1 p_2 p_3};$$

$$2. \quad \frac{\sin^2(23)}{p_2 p_3} + \frac{\sin^2(13)}{p_1 p_3} + \frac{\sin^2(12)}{p_1 p_2};$$

$$3. \quad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

Of course the last has long been used (see Treatise, p. 178). The two others have also applications.

These theorems also appear from the fact that, if a', b', c' be three conjugate semi-diameters of a quadric, then

$$a'^2 + b'^2 + c'^2,$$

$$b'^2 c'^2 \sin^2(23) + c'^2 a'^2 \sin^2(13) + a'^2 b'^2 \sin^2(12),$$

and

$$a'^2 b'^2 c'^2 \sin^2(123),$$

are each constant for any triad of conjugate diameters. If the surface be the pitch quadric, then the squares of the semi-axes are inversely proportional to the pitches, so that these three properties of coreciprocal screws in the three-system correspond exactly to properties of conjugate diameters well-known in quadric surfaces.

We can now obtain different proofs of some of the other theorems by the following process which was kindly suggested to me by Mr. Grace. We have obtained the cubic determining the stationary pitches in the three-system 1, 2, 3: let us now suppose that we seek the stationary pitches in the reciprocal system defined by 4, 5, 6. We then have the equation

$$p_\phi^3 \frac{\sin^2(456)}{p_4 p_5 p_6} - p_\phi^2 \left(\frac{\sin^2(45)}{p_4 p_5} + \frac{\sin^2(56)}{p_5 p_6} + \frac{\sin^2(46)}{p_4 p_6} \right)$$
$$+ p_\phi \left(\frac{1}{p_4} + \frac{1}{p_5} + \frac{1}{p_6} \right) - 1 = 0.$$

But we have proved (§ 14) that the roots of this equation in p_ϕ can be

no other than the roots of the former equation in p_θ, with the sign changed, and consequently

$$\frac{\sin^2(123)}{p_1 p_2 p_3} + \frac{\sin^2(456)}{p_4 p_5 p_6} = 0.$$

$$\frac{\sin^2(12)}{p_1 p_2} + \frac{\sin^2(13)}{p_1 p_3} + \frac{\sin^2(23)}{p_2 p_3} = \frac{\sin^2(45)}{p_4 p_5} + \frac{\sin^2(46)}{p_4 p_6} + \frac{\sin^2(56)}{p_5 p_6}.$$

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} + \frac{1}{p_6} = 0.$$

Thus we verify §§ 3, 4.

§ 17. Let 1, 2, 3, . . n be any system of coreciprocal screws chosen from an n-system S, where n is > 2. Let $p_1, \ldots p_n$ be the respective pitches of these n screws. Let the general symbol Σ imply every combination of three which can be formed from the n-coreciprocals; then $\Sigma \frac{\sin^2(123)}{p_1 p_2 p_3}$ has the same value for every one of the different groups of n coreciprocals which can be drawn from S.

The proof of this theorem is obtained by an induction from the results already given. Thus, in the case of $n = 3$, we have shown in the last article that $\sin^2(123) \div p_1 p_2 p_3$ was constant, while if $n = 4$ or 5 or 6, it has been shown, in each case, that $\Sigma \sin^2(123) \div p_1 p_2 p_3$ is zero. This completes the proof.

§ 18. Calculation of three invariant functions in a three-system.

By invariants, we here mean functions which are unaltered whatever be the three screws of the three-system which are introduced into them. If 1, 2, 3 be three screws defining the system, then the cubic by which the three principal pitches $p_\alpha, p_\beta, p_\gamma$ are given is

$$\begin{vmatrix} \varpi_{11} - p_\theta & \varpi_{12} - \cos(12)p_\theta & \varpi_{13} - \cos(13)p_\theta \\ \varpi_{21} - \cos(21)p_\theta & \varpi_{22} - p_\theta & \varpi_{23} - \cos(23)p_\theta \\ \varpi_{31} - \cos(31)p_\theta & \varpi_{32} - \cos(32)p_\theta & \varpi_{33} - p_\theta \end{vmatrix} = 0.$$

As the roots of this cubic must be the same from whatever three screws

of the system it be derived, we must have the following three functions the same for any three screws of the system.

The first invariant is

$$p_1 \frac{\sin^2(23)}{\sin^2(123)} + p_2 \frac{\sin^2(13)}{\sin^2(123)} + p_3 \frac{\sin^2(12)}{\sin^2(123)}$$
$$+ 2\varpi_{32} \frac{\cos(31)\cos(21) - \cos(32)}{\sin^2(123)} + 2\varpi_{13} \frac{\cos(12)\cos(32) - \cos(13)}{\sin^2(123)}$$
$$+ 2\varpi_{21} \frac{\cos(23)\cos(13) - \cos(12)}{\sin^2(123)}.$$

The second invariant is

$$\frac{p_1 p_3 + p_1 p_2 + p_2 p_3 - \varpi_{32}^2 - \varpi_{13}^2 - \varpi_{21}^2}{\sin^2(123)} + 2\cos(32) \frac{\varpi_{21}\varpi_{31} - p_1 \varpi_{23}}{\sin^2(123)}$$
$$+ 2\cos(13) \frac{\varpi_{32}\varpi_{12} - p_2 \varpi_{31}}{\sin^2(123)} + 2\cos(21) \frac{\varpi_{13}\varpi_{23} - p_3 \varpi_{12}}{\sin^2(123)}.$$

The third invariant is

$$\begin{vmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} \end{vmatrix} \div \sin^2(123).$$

Of course if the screws were coreciprocal, these expressions would reduce to those of the last article. As an illustration of the first, we may note that, if the three screws are at right angles, the expression reduces to $p_1 + p_2 + p_3$, from which we learn that the sum of the three pitches of three screws at right angles in a three-system is constant.

§ 19. A direct proof of the constancy of the third invariant.

Let θ, ϕ, ψ be any three screws, and let α, β, γ be any other triad of independent screws belonging to the three-system defined by θ, ϕ, ψ. Let the unit twist velocity on θ have component twist velocities θ_α, θ_β, θ_γ on α, β, γ respectively. Then of course θ_α, θ_β, θ_γ are the coordinates of θ with respect to α, β, γ. In like manner let ϕ_α, ϕ_β, ϕ_γ be the

coordinates of ϕ, and ψ_α, ψ_β, ψ_γ be the coordinates of ψ. Then we have first to prove the following formula

$$\begin{vmatrix} \theta_\alpha & \theta_\beta & \theta_\gamma \\ \phi_\alpha & \phi_\beta & \phi_\gamma \\ \psi_\alpha & \psi_\beta & \psi_\gamma \end{vmatrix} = \sin(\theta\phi\psi) \div \sin(\alpha\beta\gamma).$$

Let us take any other three independent screws λ, μ, ν belonging to the same system. Then θ_λ, θ_μ, θ_ν represents the coordinates of any screw θ, with respect to λ, μ, ν, and we have

$$\begin{aligned} \theta_\lambda &= \theta_\alpha\alpha_\lambda + \theta_\beta\beta_\lambda + \theta_\gamma\gamma_\lambda, \\ \theta_\mu &= \theta_\alpha\alpha_\mu + \theta_\beta\beta_\mu + \theta_\gamma\gamma_\mu, \\ \theta_\nu &= \theta_\alpha\alpha_\nu + \theta_\beta\beta_\nu + \theta_\gamma\gamma_\nu. \end{aligned}$$

Hence we easily obtain

$$\begin{vmatrix} \theta_\lambda & \theta_\mu & \theta_\nu \\ \phi_\lambda & \phi_\mu & \phi_\nu \\ \psi_\lambda & \psi_\mu & \psi_\nu \end{vmatrix} = \begin{vmatrix} \theta_\alpha & \theta_\beta & \theta_\gamma \\ \phi_\alpha & \phi_\beta & \phi_\gamma \\ \psi_\alpha & \psi_\beta & \psi_\gamma \end{vmatrix} \times \begin{vmatrix} \alpha_\lambda & \beta_\lambda & \gamma_\lambda \\ \alpha_\mu & \beta_\mu & \gamma_\mu \\ \alpha_\nu & \beta_\nu & \gamma_\nu \end{vmatrix}.$$

If now we take λ, μ, ν as the three principal screws of the system, or indeed as any three rectangular screws, we have

$$\begin{vmatrix} \theta_\lambda & \theta_\mu & \theta_\nu \\ \phi_\lambda & \phi_\mu & \phi_\nu \\ \psi_\lambda & \psi_\mu & \psi_\nu \end{vmatrix} = \sin(\theta\phi\psi), \quad \begin{vmatrix} \alpha_\lambda & \beta_\lambda & \gamma_\lambda \\ \alpha_\mu & \beta_\mu & \gamma_\mu \\ \alpha_\nu & \beta_\nu & \gamma_\nu \end{vmatrix} = \sin(\alpha\beta\gamma),$$

whence the derived formula

$$\begin{vmatrix} \theta_\alpha & \theta_\beta & \theta_\gamma \\ \phi_\alpha & \phi_\beta & \phi_\gamma \\ \psi_\alpha & \psi_\beta & \psi_\gamma \end{vmatrix} = \sin(\theta\phi\psi) \div \sin(\alpha\beta\gamma).$$

Let us take any three coreciprocals 1, 2, 3 in the three-system as screws of reference

$$\begin{aligned} \varpi_{\alpha\alpha} &= p_1\alpha_1^2 + p_2\alpha_2^2 + p_3\alpha_3^2, \\ \varpi_{\alpha\beta} &= p_1\alpha_1\beta_1 + p_2\alpha_2\beta_2 + p_3\alpha_3\beta_3; \end{aligned}$$

and therefore the determinant

$$\begin{vmatrix} \varpi_{\alpha\alpha} & \varpi_{\alpha\beta} & \varpi_{\alpha\gamma} \\ \varpi_{\beta\alpha} & \varpi_{\beta\beta} & \varpi_{\beta\gamma} \\ \varpi_{\gamma\alpha} & \varpi_{\gamma\beta} & \varpi_{\gamma\gamma} \end{vmatrix},$$

may be decomposed into the product

$$\begin{vmatrix} p_1\alpha_1 & p_2\alpha_2 & p_3\alpha_3 \\ p_1\beta_1 & p_2\beta_2 & p_3\beta_3 \\ p_1\gamma_1 & p_2\gamma_2 & p_3\gamma_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = p_1p_2p_3 \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_2 \end{vmatrix}^2,$$

but we have just proved that

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \sin(\alpha\beta\gamma) \div \sin(123);$$

and consequently we have the result sought, viz.:

$$\begin{vmatrix} \varpi_{\alpha\alpha} & \varpi_{\alpha\beta} & \varpi_{\alpha\gamma} \\ \varpi_{\beta\alpha} & \varpi_{\beta\beta} & \varpi_{\beta\gamma} \\ \varpi_{\gamma\alpha} & \varpi_{\gamma\beta} & \varpi_{\gamma\gamma} \end{vmatrix} \div \sin^2(\alpha\beta\gamma) = p_1p_2p_3 \div \sin^2(123).$$

§ 20. A theorem in determinants:

Let there be any six screws α, β, γ, δ, ϵ, ζ. Let $\alpha_1 \ldots\ldots \alpha_6$ be the coordinates of α with respect to six arbitrary screws of reference 1, 2, 3, 4, 5, 6. Let $\beta_1 \ldots\ldots \beta_6$; $\gamma_1 \ldots\ldots \gamma_6$; $\zeta_1 \ldots\ldots \xi_6$ be the corresponding coordinates of $\beta, \gamma, \ldots \zeta$ with respect to the same screws of reference. Then the determinant

$$\begin{vmatrix} \varpi_{\alpha\alpha} & \varpi_{\alpha\beta} & \varpi_{\alpha\gamma} & \varpi_{\alpha\delta} & \varpi_{\alpha\epsilon} & \varpi_{\alpha\zeta} \\ \varpi_{\beta\alpha} & \varpi_{\beta\beta} & \varpi_{\beta\gamma} & \varpi_{\beta\delta} & \varpi_{\beta\epsilon} & \varpi_{\beta\zeta} \\ \varpi_{\gamma\alpha} & \varpi_{\gamma\beta} & \varpi_{\gamma\gamma} & \varpi_{\gamma\delta} & \varpi_{\gamma\epsilon} & \varpi_{\gamma\zeta} \\ \varpi_{\delta\alpha} & \varpi_{\delta\beta} & \varpi_{\delta\gamma} & \varpi_{\delta\delta} & \varpi_{\delta\epsilon} & \varpi_{\delta\zeta} \\ \varpi_{\epsilon\alpha} & \varpi_{\epsilon\beta} & \varpi_{\epsilon\gamma} & \varpi_{\epsilon\delta} & \varpi_{\epsilon\epsilon} & \varpi_{\epsilon\zeta} \\ \varpi_{\zeta\alpha} & \varpi_{\zeta\beta} & \varpi_{\zeta\gamma} & \varpi_{\zeta\delta} & \varpi_{\zeta\epsilon} & \varpi_{\zeta\zeta} \end{vmatrix} \text{ i.}$$

is equal to the product of the square of the determinant

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 \end{vmatrix} \quad \text{ii.}$$

by the determinant

$$\begin{vmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} & \varpi_{14} & \varpi_{15} & \varpi_{16} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} & \varpi_{24} & \varpi_{25} & \varpi_{26} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \varpi_{41} & \varpi_{42} & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \varpi_{51} & \varpi_{52} & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \varpi_{61} & \varpi_{62} & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix} \quad \text{iii.}$$

To prove this we multiply the two last determinants together, the two first elements in the product are

$$\varpi_{11}\alpha_1 + \varpi_{12}\alpha_2 + \varpi_{13}\alpha_3 + \varpi_{14}\alpha_4 + \varpi_{15}\alpha_5 + \varpi_{16}\alpha_6,$$
$$\varpi_{11}\beta_1 + \varpi_{12}\beta_2 + \varpi_{13}\beta_3 + \varpi_{14}\beta_4 + \varpi_{15}\beta_5 + \varpi_{16}\beta_6;$$

but these expressions may be concisely replaced by $\varpi_{\alpha 1}$ and $\varpi_{\beta 1}$ respectively. It follows that the product of the two last determinants is

$$\begin{vmatrix} \varpi_{\alpha 1} & \varpi_{\alpha 2} & \varpi_{\alpha 3} & \varpi_{\alpha 4} & \varpi_{\alpha 5} & \varpi_{\alpha 6} \\ \varpi_{\beta 1} & \varpi_{\beta 2} & \varpi_{\beta 3} & \varpi_{\beta 4} & \varpi_{\beta 5} & \varpi_{\beta 6} \\ \varpi_{\gamma 1} & \varpi_{\gamma 2} & \varpi_{\gamma 3} & \varpi_{\gamma 4} & \varpi_{\gamma 5} & \varpi_{\gamma 6} \\ \varpi_{\delta 1} & \varpi_{\delta 2} & \varpi_{\delta 3} & \varpi_{\delta 4} & \varpi_{\delta 5} & \varpi_{\delta 6} \\ \varpi_{\epsilon 1} & \varpi_{\epsilon 2} & \varpi_{\epsilon 3} & \varpi_{\epsilon 4} & \varpi_{\epsilon 5} & \varpi_{\epsilon 6} \\ \varpi_{\zeta 1} & \varpi_{\zeta 2} & \varpi_{\zeta 3} & \varpi_{\zeta 4} & \varpi_{\zeta 5} & \varpi_{\zeta 6} \end{vmatrix}.$$

If we multiply this again by ii. we have, as the two first elements in the product,

$$\alpha_1\varpi_{\alpha 1} + \alpha_2\varpi_{\alpha 2} + \alpha_3\varpi_{\alpha 3} + \alpha_4\varpi_{\alpha 4} + \alpha_5\varpi_{\alpha 5} + \alpha_6\varpi_{\alpha 6},$$

and

$$\alpha_1\varpi_{\beta 1} + \alpha_2\varpi_{\beta 2} + \alpha_3\varpi_{\beta 3} + \alpha_4\varpi_{\beta 4} + \alpha_5\varpi_{\beta 5} + \alpha_6\varpi_{\beta 6};$$

but the first of them is $\varpi_{\alpha\alpha}$ and the second is $\varpi_{\alpha\beta}$. In like manner for the other terms, and the theorem has thus been proved.

§ 21. Let $\alpha_1, \ldots \alpha_6$ be the six coordinates of a screw α of pitch p_α with respect to any six screws of reference 1, 2, 3, 4, 5, 6, whether coreciprocal or not. It is required to determine $\theta_1 \ldots \theta_6$ the coordinates of a screw θ, of pitch p_θ, which lies on the same axis as α.

The solution of this problem is well known (Treatise, pp. 41, 109) in the case where the screws of reference are coreciprocal. It is the more general case where the screws of reference are any independent screws whatever, that we are now to discuss.

Let θ and ϕ be any two screws, then the condition that they shall be reciprocal is easily seen to be (Treatise, p. 33),

$$p_1\theta_1\phi_1 + p_2\theta_2\phi_2 + p_3\theta_3\phi_3 + p_4\theta_4\phi_4 + p_5\theta_5\phi_5 + p_6\theta_6\phi_6$$
$$+ \varpi_{12}(\theta_1\phi_2 + \theta_2\phi_1) + \varpi_3(\theta_1\phi_3 + \theta_3\phi_1) + \&c. = 0,$$

which may be written

$$\phi_1\frac{dp_\theta}{d\theta_1} + \phi_2\frac{dp_\theta}{d\theta_2}\ldots \qquad + \phi_6\frac{dp_\theta}{d\theta_6} = 0.$$

If the six coordinates of ϕ satisfy the linear equation

$$A_1\phi_1 + \ldots \qquad + A_6\phi_6 = 0,$$

then every screw ϕ will be reciprocal to the screw θ whose coordinates are given by

$$A_1 = \frac{dp_\theta}{d\theta_1}\ldots \qquad A_6 = \frac{dp_\theta}{d\theta_6}.$$

Let $U = 0$ be an equation in the second degree in the screw coordinates of α, then the five-system polar to α, with respect to U, has for its equation (Treatise, p. 238),

$$\phi_1\frac{dU}{d\alpha_1}\ldots \qquad + \phi_6\frac{dU_\alpha}{d\alpha_6} = 0.$$

If, therefore, θ be the screw reciprocal to the five-system, we have

$$\frac{dp_\theta}{d\theta_1} = \frac{dU_\alpha}{d\alpha_1}\ldots \qquad \frac{dp_\theta}{d\theta_6} = \frac{dU_\alpha}{d\alpha_6},$$

and we obtain accordingly the result which may be thus stated.

Let $U_\eta = 0$ be an homogeneous equation in the second degree in the coordinates of a screw η with regard to any six screws of reference which do not themselves belong to a five-system. If θ be the polar screw of any screw α with respect to the quadratic five-system (Treatise, p. 236), expressed by $U_\eta = 0$, then

$$\frac{dp_\theta}{d\theta_1} = \frac{dU_\alpha}{d\alpha_1}; \quad \ldots \quad \frac{dp_\theta}{d\theta_2} = \frac{dU_\alpha}{d\alpha_2}; \quad \ldots \quad \frac{dp_\theta}{d\theta_6} = \frac{dU_\alpha}{d\alpha_6};$$

we thus obtain six linear equations connecting the coordinates of each screw and its polar with regard to this particular quadratic five-system.

Let us now assume for U_η the form $p_\eta - kR_\eta$. The quadratic five-system will then mean no more than the aggregate of all possible screws with the pitch k. Draw any cylindroid C through α, then two screws of pitch k, i.e. two screws of the system $U_\eta = 0$ must, in general, be found on the cylindroid. The harmonic conjugate of α with respect to these two screws of pitch k, and which lies on the cylindroid C, will belong to the five-system polar to α with reference to the quadratic five-system $U_\eta = 0$ (Treatise, p. 239). In the construction of the screw polar to α, the particular screws of reference belonging to the quadratic five-system are not involved. The symmetry of the case therefore requires that the polar screw of α, with respect to this particular quadratic five-system, shall be coincident in situation with α. We have therefore for the coordinates of the screw θ, the six equations

$$\left.\begin{array}{l} h\dfrac{dp_\theta}{d\theta_1} = \dfrac{dp_\alpha}{d\alpha_1} - k\dfrac{dR}{d\alpha_1}, \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ h\dfrac{dp_\theta}{d\theta_6} = \dfrac{dp_\alpha}{d\alpha_6} - k\dfrac{dR}{d\alpha_6}; \end{array}\right\} \quad \text{i.}$$

multiplying these equations by $\theta_1 \ldots \theta_6$, adding them and observing that as α and θ must be coincident in situation,

$$2\varpi_{\alpha\theta} = \theta_1\frac{dp_\alpha}{d\alpha_1} + \ldots + \theta_6\frac{dp_\alpha}{d\alpha_6} = p_\alpha + p_\theta,$$

$$\theta_1\frac{dR}{d\alpha_1} + \ldots + \theta_6\frac{dR}{d\alpha_6} = 2\cos(\alpha\theta) = 2,$$

we obtain

$$2hp_\theta = p_\alpha + p_\theta - 2k.$$

If now we multiply equation i. by $\alpha_1 \ldots \alpha_6$, respectively, and add, we obtain

$$h(p_\alpha + p_\theta) = 2p_\alpha - 2k,$$

whence it appears that $h = 1$, $k = \frac{p_\alpha - p_\theta}{2}$, and we learn that $\theta_1 \ldots \theta_6$, the coordinates of the screw coincident in situation with α, but with pitch p_θ, are given by

$$\begin{aligned} \frac{dp_\theta}{d\theta_1} &= \frac{dp_\alpha}{d\alpha_1} - \frac{p_\alpha - p_\theta}{2}\frac{dR}{d\alpha_1}, \\ &\vdots \\ \frac{dp_\theta}{d\theta_6} &= \frac{dp_\alpha}{d\alpha_6} - \frac{p_\alpha - p_\theta}{2}\frac{dR}{d\alpha_6}. \end{aligned}$$

We now solve these equations for $\theta_1 \ldots \theta_6$. If we substitute the general expressions for p_θ and p_α, the first of the equations becomes

$$\begin{aligned} &2\varpi_{11}\theta_1 + 2\varpi_{12}\theta_2 + 2\varpi_{13}\theta_3 + 2\varpi_{14}\theta_4 + 2\varpi_{15}\theta_5 + 2\varpi_{16}\theta_6 \\ &= 2\varpi_{11}\alpha_1 + 2\varpi_{12}\alpha_2 + 2\varpi_{13}\alpha_3 + 2\varpi_{14}\alpha_4 + 2\varpi_{15}\alpha_5 + 2\varpi_{16}\alpha_6 - \frac{p_\alpha - p_\theta}{2}\frac{dR}{d\alpha_1}. \end{aligned}$$

There are five other similar equations, and if we make

$$\Delta = \begin{vmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} & \varpi_{14} & \varpi_{15} & \varpi_{16} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} & \varpi_{24} & \varpi_{25} & \varpi_{26} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \varpi_{41} & \varpi_{42} & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \varpi_{51} & \varpi_{52} & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \varpi_{61} & \varpi_{62} & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix},$$

we obtain after a few reductions, and remembering that $\frac{dR}{d\alpha_1} = 2\cos(\alpha 1)$, &c.,

$$\theta_1 = \alpha_1 - \frac{p_\alpha - p_\theta}{2\Delta}\begin{vmatrix} \cos(\alpha 1) & \varpi_{12} & \varpi_{13} & \varpi_{14} & \varpi_{15} & \varpi_{16} \\ \cos(\alpha 2) & \varpi_{22} & \varpi_{23} & \varpi_{24} & \varpi_{25} & \varpi_{26} \\ \cos(\alpha 3) & \varpi_{32} & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \cos(\alpha 4) & \varpi_{42} & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \cos(\alpha 5) & \varpi_{52} & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \cos(\alpha 6) & \varpi_{62} & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix},$$

with similar values for the other coordinates.

We are thus led to the following solution of the problem in its most general form:

Let α be a screw of any pitch p_α whose coordinates are $\alpha_1, \ldots \alpha_6$ with respect to any six independent screws of reference 1, 2, 3, 4, 5, 6; then a screw θ whose axis is coincident with the axis of α, but which has a pitch p_θ, will have as its coordinates, with respect to the same six screws of reference, $\theta_1 \ldots \theta_6$, where

$$\theta_1 = \alpha_1 + \frac{p_\alpha - p_\theta}{2\Delta}\frac{dV}{d\epsilon_1}, \qquad \theta_2 = \alpha_2 + \frac{p_\alpha - p_\theta}{2\Delta}\frac{dV}{d\epsilon_2},$$

$$\theta_3 = \alpha_3 + \frac{p_\alpha - p_\theta}{2\Delta}\frac{dV}{d\epsilon_3}, \qquad \theta_4 = \alpha_4 + \frac{p_\alpha - p_\theta}{2\Delta}\frac{dV}{d\epsilon_4},$$

$$\theta_5 = \alpha_5 + \frac{p_\alpha - p_\theta}{2\Delta}\frac{dV}{d\epsilon_5}, \qquad \theta_6 = \alpha_6 + \frac{p_\alpha - p_\theta}{2\Delta}\frac{dV}{d\epsilon_6},$$

in which V is the determinant:

$$\begin{vmatrix} 0 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \\ \cos(\alpha 1) & \varpi_{11} & \varpi_{12} & \varpi_{13} & \varpi_{14} & \varpi_{15} & \varpi_{16} \\ \cos(\alpha 2) & \varpi_{21} & \varpi_{22} & \varpi_{23} & \varpi_{24} & \varpi_{25} & \varpi_{26} \\ \cos(\alpha 3) & \varpi_{31} & \varpi_{32} & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \cos(\alpha 4) & \varpi_{41} & \varpi_{42} & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \cos(\alpha 5) & \varpi_{51} & \varpi_{52} & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \cos(\alpha 6) & \varpi_{61} & \varpi_{62} & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix},$$

and in which Δ is the determinant remaining when the first row and the first column have been erased from V.

§ 22. General theorem connecting the coordinates of any two screws whatever referred to any six screws of reference which do not themselves belong to a five-system but are otherwise unrestricted.

Let α and β be any two screws. Let ϕ be the angle between α and β. Let ρ be a screw on the cylindroid $(\alpha\beta)$ and separated from α only by the small angle ϵ.

Let the unit twist velocity on ρ be resolved into component twist velocities on α and β. Then, by the fundamental property of the cylindroid (Treatise, p. 21), these components are

$$\text{on } \alpha \qquad \frac{\sin(\phi-\epsilon)}{\sin\phi} = 1 - \epsilon\frac{\cos\phi}{\sin\phi},$$

$$\text{on } \beta \qquad \frac{\sin\epsilon}{\sin\phi} = \frac{\epsilon}{\sin\phi}.$$

But the component twist velocities $\rho_1 \ldots \rho_n$ may be obtained either by resolving the unit twist velocity on ρ directly, or by first decomposing, as we have done, this twist velocity on ρ into twist velocities on α and β, and then summing on the screws of reference the components obtained by resolving the twist velocities on α and β separately.

We thus obtain

$$\begin{aligned} \rho_1 &= \alpha_1 + \epsilon\operatorname{cosec}\phi(\beta_1 - \alpha_1\cos\phi), \\ &\vdots \qquad\qquad\qquad \vdots \\ \rho_n &= \alpha_n + \epsilon\operatorname{cosec}\phi(\beta_n - \alpha_n\cos\phi). \end{aligned}$$

But these n quantities $\rho_1 \ldots \rho_n$ must make (Treatise, p. 35),

$$R_\rho = \Sigma\rho_1^2 + 2\Sigma\rho_1\rho_2\cos(12) = 1.$$

If we substitute the values for ρ_1, &c., we obtain

$$R_\alpha + \epsilon\operatorname{cosec}\phi\left[(\beta_1 - \alpha_1\cos\phi)\frac{dR}{d\alpha_1} + \ldots + (\beta_n - \alpha_n\cos\phi)\frac{dR}{d\alpha_n}\right] = 1.$$

The quantity inside the bracket must be zero, and as

$$\alpha_1\frac{dR}{d\alpha_1} + \ldots + \alpha_n\frac{dR}{d\alpha_n} = 2R = 2,$$

we obtain

$$\cos\phi = \tfrac{1}{2}\Sigma\beta_1\frac{dR}{d\alpha_1} = \tfrac{1}{2}\Sigma\alpha_1\frac{dR}{d\beta_1}.$$

This important expression was already well known (Treatise, p. 276), but the above method of obtaining it is perhaps the most direct.

As $\frac{1}{2}\frac{dR}{d\alpha_1} = \cos(\alpha 1)$, we may express this in the form

$$\cos\phi = \Sigma\beta_1\cos(\alpha 1) = \Sigma\alpha_1\cos(\beta 1).$$

As the cosine of the angle between two screws is not altered by merely adding to the pitch of one of them, it is plain from the last article that

$$\Sigma \alpha_1 \cos(\beta 1) = \Sigma\left(\alpha_1 + \frac{p_\alpha - p_\theta}{2\Delta}\frac{dV}{d\epsilon_1}\right)\cos(\beta 1);$$

and consequently,

$$\frac{dV}{d\epsilon_1}\cos(\beta 1) + \frac{dV}{d\epsilon_2}\cos(\beta 2) + \ldots + \frac{dV}{d\epsilon_6}\cos(\beta 6) = 0.$$

Observing the value of V as there given, this equation means that the following determinant is zero:

$$\begin{vmatrix} 0 & \cos(\beta 1) & \cos(\beta 2) & \cos(\beta 3) & \cos(\beta 4) & \cos(\beta 5) & \cos(\beta 6) \\ \cos(\alpha 1) & \varpi_{11} & \varpi_{12} & \varpi_{13} & \varpi_{14} & \varpi_{15} & \varpi_{16} \\ \cos(\alpha 2) & \varpi_{21} & \varpi_{22} & \varpi_{23} & \varpi_{24} & \varpi_{25} & \varpi_{26} \\ \cos(\alpha 3) & \varpi_{31} & \varpi_{32} & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \cos(\alpha 4) & \varpi_{41} & \varpi_{42} & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \cos(\alpha 5) & \varpi_{51} & \varpi_{52} & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \cos(\alpha 6) & \varpi_{61} & \varpi_{62} & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix} \quad \text{i.}$$

This is a remarkable equation, connecting, as it does, any two screws whatever with the six screws of reference, which are themselves entirely arbitrary.

In the case of a coreciprocal system, the theorem degrades to a formula which has been already given in Treatise, p. 288, namely,

$$\frac{\cos(\alpha 1)\cos(\beta 1)}{p_1} + \frac{\cos(\alpha 2)\cos(\beta 2)}{p_2} + \&c., \; + \frac{\cos(\alpha 6)\cos(\beta 6)}{p_6} = 0,$$

whatever α and β may be.

It may be noted that there are certain other determinants involving the coordinates of two screws, which must also vanish. They are obtained as follows:—The general theorem expressed in formula i must remain true if the pitch of one of the screws of reference, say for

example p_1, be increased by any quantity, suppose $2x$. Such a change would alter ϖ_{12} into $\varpi_{12} + x\cos(12)$, and therefore whatever be the value of x, the following determinant must vanish:—

$$\begin{vmatrix} 0 & \cos(\alpha 1) & \cos(\alpha 2) & \cos(\alpha 3) & \cos(\alpha 4) & \cos(\alpha 5) & \cos(\alpha 6) \\ \cos(\beta 1) & \varpi_{11} + x\cos(11) & \varpi_{12} + x\cos(12) & \varpi_{13} + x\cos(13) & \varpi_{14} + x\cos(14) & \varpi_{15} + x\cos(15) & \varpi_{16} + x\cos(16) \\ \cos(\beta 2) & \varpi_{21} + x\cos(21) & \varpi_{22} & \varpi_{23} & \varpi_{24} & \varpi_{25} & \varpi_{26} \\ \cos(\beta 3) & \varpi_{31} + x\cos(31) & \varpi_{32} & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \cos(\beta 4) & \varpi_{41} + x\cos(41) & \varpi_{42} & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \cos(\beta 5) & \varpi_{51} + x\cos(51) & \varpi_{52} & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \cos(\beta 6) & \varpi_{61} + x\cos(61) & \varpi_{62} & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix} \quad \text{ii.}$$

when this is expanded it contains terms in x^2 and x, whereof each must be separately zero. To find the coefficient of x^2, we observe that $\cos(\alpha 1)$ in the first row cannot be a factor of x^2. Therefore, in finding the coefficient of x^2, we may make $\cos(\alpha 1)$ zero. In like manner, $\cos(\beta 1)$ may be made zero. It is also plain that $\varpi_{11}, \varpi_{12}, \varpi_{13}, \varpi_{14}, \varpi_{15}, \varpi_{16}$ can never multiply x^2. In computing the coefficient of x^2, we may make them all zero; further, $\cos(11)$ in the second term of the second row cannot multiply x^2, and it may also be made zero. We thus see that the evanescence of the coefficient of x^2 gives the following equation:—

$$\begin{vmatrix} 0 & 0 & \cos(\alpha 2) & \cos(\alpha 3) & \cos(\alpha 4) & \cos(\alpha 5) & \cos(\alpha 6) \\ 0 & 0 & \cos(12) & \cos(13) & \cos(14) & \cos(15) & \cos(16) \\ \cos(\beta 2) & \cos(21) & \varpi_{22} & \varpi_{23} & \varpi_{24} & \varpi_{25} & \varpi_{26} \\ \cos(\beta 3) & \cos(31) & \varpi_{32} & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \cos(\beta 4) & \cos(41) & \varpi_{42} & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \cos(\beta 5) & \cos(51) & \varpi_{52} & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \cos(\beta 6) & \cos(61) & \varpi_{62} & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix} = 0. \quad \text{iii.}$$

As this is perfectly general, it must remain true if ϖ_{22}, that is p_2, be increased by x, and again equating the coefficient of x^2 to zero, we have

$$\begin{vmatrix} 0 & 0 & 0 & \cos(\alpha 3) & \cos(\alpha 4) & \cos(\alpha 5) & \cos(\alpha 6) \\ 0 & 0 & 0 & \cos(13) & \cos(14) & \cos(15) & \cos(16) \\ 0 & 0 & 0 & \cos(23) & \cos(24) & \cos(25) & \cos(26) \\ \cos(\beta 3) & \cos(31) & \cos(32) & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \cos(\beta 4) & \cos(41) & \cos(42) & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \cos(\beta 5) & \cos(51) & \cos(52) & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \cos(\beta 6) & \cos(61) & \cos(62) & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix} = 0. \quad \text{iiii.}$$

The determinant iii leads to a theorem connecting the screws of a five-system, which is worthy of attention. Let us suppose that the five screws 2, 3, 4, 5, 6 are coreciprocal. As 1 is arbitrary, we shall designate it as θ, and replacing 2, 3, 4, 5, 6 by 1, 2, 3, 4, 5 respectively, and re-arranging the determinant, we obtain

$$\begin{vmatrix} p_1 & 0 & 0 & 0 & 0 & \cos(\beta 1) & \cos(\theta 1) \\ 0 & p_2 & 0 & 0 & 0 & \cos(\beta 2) & \cos(\theta 2) \\ 0 & 0 & p_3 & 0 & 0 & \cos(\beta 3) & \cos(\theta 3) \\ 0 & 0 & 0 & p_4 & 0 & \cos(\beta 4) & \cos(\theta 4) \\ 0 & 0 & 0 & 0 & p_5 & \cos(\beta 5) & \cos(\theta 5) \\ \cos(\alpha 1) & \cos(\alpha 2) & \cos(\alpha 3) & \cos(\alpha 4) & \cos(\alpha 5) & 0 & 0 \\ \cos(\theta 1) & \cos(\theta 2) & \cos(\theta 3) & \cos(\theta 4) & \cos(\theta 5) & 0 & 0 \end{vmatrix} = 0. \quad \text{iii.}$$

If we expand this, it becomes

$$\Sigma \frac{1}{p_1 p_2} \begin{vmatrix} \cos(\alpha 1) & \cos(\alpha 2) \\ \cos(\theta 1) & \cos(\theta 2) \end{vmatrix} \times \begin{vmatrix} \cos(\beta 1) & \cos(\beta 2) \\ \cos(\theta 1) & \cos(\theta 2) \end{vmatrix} = 0.$$

Let us take α and β as two rectangular directions, and let θ be normal to the plane of α and β. If then we denote by $(12)_\alpha$ and $(12)_\beta$ the cosines of the angles made by the normal to 1 and 2 with α and β

respectively, we obtain the following result which must be true for the five coreciprocal screws, whatever α and β may be,

$$\Sigma \frac{(12)_\alpha (12)_\beta \sin^2(12)}{p_1 p_2} = 0.$$

There can be various other formulæ, for α, β and θ are quite arbitrary. Let β coincide with α. Then, whatever θ and α may be, we have, in the five-system,

$$\Sigma \frac{1}{p_1 p_2}\Big(\cos(\alpha 1)\cos(\theta 2) - \cos(\alpha 2)\cos(\theta 1)\Big)^2 = 0.$$

Another deduction may be made from determinant iii. Let α, β and θ be all perpendicular to 1, which requires that the plane of α, β, θ shall be parallel to the axis of the cylindroid reciprocal to 2, 3, 4, 5. If we further make α and β coincident, and write 1, 2, 3, 4 instead of 2, 3, 4, 5, we obtain the following property of four coreciprocal screws 1, 2, 3, 4 :—

$$\Sigma \frac{1}{p_1 p_2}\Big(\cos(\alpha 1)\cos(\theta 2) - \cos(\alpha 2)\cos(\theta 1)\Big)^2 = 0,$$

where α and θ are any two screws whose common perpendicular is normal to the axis of the cylindroid reciprocal to 1, 2, 3, 4.

We can also derive from iiii a general property of the four-system. Let us suppose the four screws 3, 4, 5, 6 are coreciprocal. As 1 and 2 are now both quite arbitrary, we shall designate them as θ and ϕ, and replacing 3, 4, 5, 6 by 1, 2, 3, 4 and re-arranging the determinant, it becomes the following:—

$$0 = \begin{vmatrix} p_1 & 0 & 0 & 0 & \cos(\beta 1) & \cos(\theta 1) & \cos(\phi 1) \\ 0 & p_2 & 0 & 0 & \cos(\beta 2) & \cos(\theta 2) & \cos(\phi 2) \\ 0 & 0 & p_3 & 0 & \cos(\beta 3) & \cos(\theta 3) & \cos(\phi 3) \\ 0 & 0 & 0 & p_4 & \cos(\beta 4) & \cos(\theta 4) & \cos(\phi 4) \\ \cos(\alpha 1) & \cos(\alpha 2) & \cos(\alpha 3) & \cos(\alpha 4) & 0 & 0 & 0 \\ \cos(\theta 1) & \cos(\theta 2) & \cos(\theta 3) & \cos(\theta 4) & 0 & 0 & 0 \\ \cos(\phi 1) & \cos(\phi 2) & \cos(\phi 3) & \cos(\phi 4) & 0 & 0 & 0 \end{vmatrix}$$

As a particular case, we may take α and β as identical with another screw ψ, and hence we have the following theorem about a set of four coreciprocal screws.

Let 1, 2, 3, 4 be any four coreciprocal screws, and let θ, ϕ, ψ be any other three screws, then

$$\Sigma\, p_1 \Omega_1^2 = 0.$$

$$\text{where } \Omega_1 = \begin{vmatrix} \cos(\theta 2) & \cos(\theta 3) & \cos(\theta 4) \\ \cos(\phi 2) & \cos(\phi 3) & \cos(\phi 4) \\ \cos(\psi 2) & \cos(\psi 3) & \cos(\psi 4) \end{vmatrix}$$

If θ, ϕ, ψ be any three rectangular axes, then $\Omega_1 = \sin(123)$, and we are again conducted to the known theorem of § 5, of which the present may be regarded as a generalization. Indeed it is easy to reduce this theorem to § 5 even without the assumption that θ, ϕ, and ψ are rectangular. The possible deductions from these general formulæ seem very numerous.

We may note another verification of the formulæ as follows:—

Let us suppose that the pitch of a screw α is increased by a small quantity x, then we have (§ 22),

$$p_\alpha + x = p_\alpha - \frac{x}{2\Delta}\frac{dp_\alpha}{d\alpha_1}\cdot\frac{dV}{d\epsilon_1} - \ldots - \frac{x}{2\Delta}\frac{dp_\alpha}{d\alpha_6}\cdot\frac{dV}{d\epsilon_6};$$

or,

$$2\Delta = -\frac{dp_\alpha}{d\alpha_1}\cdot\frac{dV}{d\epsilon_1} - \ldots - \frac{dp_\alpha}{d\alpha_6}\cdot\frac{dV}{d\epsilon_6};$$

or,

$$\Delta = -\begin{vmatrix} 0 & \frac{1}{2}\frac{dp_\alpha}{d\alpha_1} & \frac{1}{2}\frac{dp_\alpha}{d\alpha_2} & \frac{1}{2}\frac{dp_\alpha}{d\alpha_3} & \frac{1}{2}\frac{dp_\alpha}{d\alpha_4} & \frac{1}{2}\frac{dp_\alpha}{d\alpha_5} & \frac{1}{2}\frac{dp_\alpha}{d\alpha_6} \\ \cos(\alpha 1) & \varpi_{11} & \varpi_{12} & \varpi_{13} & \varpi_{14} & \varpi_{15} & \varpi_{16} \\ \cos(\alpha 2) & \varpi_{21} & \varpi_{22} & \varpi_{23} & \varpi_{24} & \varpi_{25} & \varpi_{26} \\ \cos(\alpha 3) & \varpi_{31} & \varpi_{32} & \varpi_{33} & \varpi_{34} & \varpi_{35} & \varpi_{36} \\ \cos(\alpha 4) & \varpi_{41} & \varpi_{42} & \varpi_{43} & \varpi_{44} & \varpi_{45} & \varpi_{46} \\ \cos(\alpha 5) & \varpi_{51} & \varpi_{52} & \varpi_{53} & \varpi_{54} & \varpi_{55} & \varpi_{56} \\ \cos(\alpha 6) & \varpi_{61} & \varpi_{62} & \varpi_{63} & \varpi_{64} & \varpi_{65} & \varpi_{66} \end{vmatrix}.$$

But this is obviously true, for

$$p_a = \varpi_{11} a_1^2 + \ldots + \varpi_{66} a_6^2 + 2\varpi_{12} a_1 a_2 + \ldots$$
$$l = a_1 \cos(a1) + a_2 \cos(a2) \ldots + a_6 \cos(a6).$$

Multiplying the second row of the determinant by a_1, the third by a_2, and the last by a_6, and subtracting the sum of the row thus treated from the first row, the truth of the formula is evident.

§ 23. Let 1, 2, 3, 4 be four coreciprocal screws of pitches p_1, p_2, p_3, p_4, and let $(\theta 1)$, $(\theta 2)$, $(\theta 3)$, $(\theta 4)$ be the angles made with 1, 2, 3, 4 by the axis of the cylindroid reciprocal to 1, 2, 3, 4. Then we have

$$\frac{\cos(\theta 1)}{p_1 \sin(234)} = \pm \frac{\cos(\theta 2)}{p_2 \sin(134)} = \pm \frac{\cos(\theta 3)}{p_3 \sin(124)} = \pm \frac{\cos(\theta 4)}{p_4 \sin(123)}.$$

This theorem may be regarded as an illustration of the many properties that may be deduced from the fundamental formula (Treatise, p. 288), that if $(a1)$, $(a2)$, ... $(a6)$ be the six angles made by a screw a with six coreciprocals, and if $(b1)$, $(b2)$, ... $(b6)$ be the similar angles for a screw θ, then

$$\frac{\cos(a\,1)\cos(\theta\,1)}{p_1} + \ldots + \frac{\cos(a\,6)\cos(\theta\,6)}{p_6} = 0.$$

The reciprocal cylindroid to 1, 2, 3, 4 is determined by 5 and 6. If therefore we take as θ a screw perpendicular to 5 and 6, we shall have $\cos(\theta 5) = 0$, $\cos(\theta 6) = 0$. If, further, we write the formula four times over, taking for a the screws 1, 2, 3, 4 in succession, we have

$$\frac{\cos(\theta 1)}{p_1} + \frac{\cos(\theta 2)\cos(12)}{p_2} + \frac{\cos(\theta 3)\cos(13)}{p_3} + \frac{\cos(\theta 4)\cos(14)}{p_4} = 0,$$
$$\frac{\cos(\theta 1)\cos(21)}{p_1} + \frac{\cos(\theta 2)}{p_2} + \frac{\cos(\theta 3)\cos(23)}{p_3} + \frac{\cos(\theta 4)\cos(24)}{p_4} = 0,$$
$$\frac{\cos(\theta 1)\cos(31)}{p_1} + \frac{\cos(\theta 2)\cos(32)}{p_2} + \frac{\cos(\theta 3)}{p_3} + \frac{\cos(\theta 4)\cos(34)}{p_4} = 0,$$
$$\frac{\cos(\theta 1)\cos(41)}{p_1} + \frac{\cos(\theta 2)\cos(42)}{p_2} + \frac{\cos(\theta 3)\cos(43)}{p_3} + \frac{\cos(\theta 4)}{p_4} = 0.$$

Regarding these as four equations in $\frac{\cos(\theta 1)}{p_1}$, $\frac{\cos(\theta 2)}{p_2}$, &c., we have $\frac{\cos^2(\theta 1)}{p_1^2}$ proportional to

$$\begin{vmatrix} 1 & \cos(23) & \cos(24) \\ \cos(23) & 1 & \cos(34) \\ \cos(24) & \cos(34) & 1 \end{vmatrix},$$

which is $\sin^2(234)$, so that the theorem is proved.

We may now deduce from this the fundamental theorem of § 5, for we know (Treatise, p. 287) that, for any screw θ, we have

$$\frac{\cos^2(\theta 1)}{p_1} \ldots + \frac{\cos^2(\theta 6)}{p_6} = 0.$$

In the present case $\cos(\theta 5)$ and $\cos(\theta 6)$ are both zero, so that substituting for $\frac{\cos^2(\theta 1)}{p_1}$ its value, $p_1 \sin^2(234)$, we have again

$$p_1 \sin^2(234) + p_2 \sin^2(134) + p_3 \sin^2(124) + p_4 \sin^2(123) = 0. \qquad \text{i.}$$

We know that four forces on the four rays of a pencil parallel to 1, 2, 3, 4 will equilibrate if the forces be respectively proportional to $\sin(234)$, $\sin(134)$, &c. Resolving them along any direction θ, these forces must equilibrate, and accordingly,

$$\cos(\theta 1)\sin(234) + \cos(\theta 2)\sin(134) + \cos(\theta 3)\sin(124) + \cos(\theta 4)\sin(123) = 0. \quad \text{ii.}$$

As there are two disposables in the choice of θ, we can choose θ so that

$$\frac{p_1 \sin(234)}{\cos(\theta 1)} = \frac{p_2 \sin(134)}{\cos(\theta 2)} = \frac{p_3 \sin(124)}{\cos(\theta 3)},$$

and then the comparison of the formulæ i. and ii. will show that $\frac{p_4 \sin(123)}{\cos(\theta 4)}$ must be also equal to the other similar quantities.

We may also note the following point in this connexion. If four wrenches have a couple as their resultant, then this couple is composed

of two special component couples. There is first the resultant of the four couples obtained by multiplying the intensity of each wrench into the pitch of the corresponding screw. The second component couple arises from the transference of the forces to a point where they neutralize. The former of these components has an axis parallel to that of the reciprocal cylindroid.

For if θ be the axis of the reciprocal cylindroid, then a couple of which θ is the axis will have components proportional to $\cos(\theta 1)$, $\cos(\theta 2)$, $\cos(\theta 3)$, $\cos(\theta 4)$ on the four screws 1, 2, 3, 4. But we have seen that these are proportional to $p_1 \sin(234)$, $p_2 \sin 134)$, &c., and since $\sin(234)$, $\sin(134)$, &c., are the intensities of four neutralizing forces on 1, 2, 3, 4, the theorem is proved.

§ 24. Let θ_1, θ_2, θ_3, θ_4 be amplitudes of twists about four given screws 1, 2, 3, 4, respectively, and let their amplitudes be such that the joint effect of the four twists is no more than a translation through a distance h in the direction parallel to the screw ρ. If then α be any screw whatever, we have

$$\theta_1 \varpi_{1\alpha} + \theta_2 \varpi_{2\alpha} + \theta_3 \varpi_{3\alpha} + \theta_4 \varpi_{4\alpha} = \tfrac{1}{2} h \cos(\alpha\rho).$$

The resultant of the four twists can be neutralized by a translation conveying the whole system parallel to ρ and through a distance $-h$. Each of the five movements referred to will accomplish a certain quantity of work against a wrench of intensity α'' on a screw α. But as these movements neutralize, the algebraical sum of these quantities of work must be zero. But the work done by a twist of amplitude θ_1 about θ against the wrench α'' on α is $2\theta_1 \alpha'' \varpi_{\alpha 1}$, while the displacement h does the work $-h\alpha'' \cos(\alpha\rho)$, whence

$$2\theta_1 \alpha'' \varpi_{\alpha 1} + 2\theta_2 \alpha'' \varpi_{\alpha 2} + 2\theta_3 \alpha'' \varpi_{\alpha 3} + 2\theta_4 \alpha'' \varpi_{\alpha 4} - h\alpha'' \cos(\alpha\rho) = 0,$$

and, dividing by $2\alpha''$, we obtain the desired formula.

If we take 1, 2, 3, 4 in succession for α, and if we observe that the

values of θ_1, &c., are $\sin(234)$, &c., we obtain the following formulæ where m is a common factor:

$$m\cos(\rho 1) = \sin(234)\varpi_{11} + \sin(134)\varpi_{21} + \sin(124)\varpi_{31} + \sin(123)\varpi_{41},$$
$$m\cos(\rho 2) = \sin(234)\varpi_{21} + \sin(134)\varpi_{22} + \sin(124)\varpi_{23} + \sin(123)\varpi_{42},$$
$$m\cos(\rho 3) = \sin(234)\varpi_{31} + \sin(134)\varpi_{32} + \sin(124)\varpi_{33} + \sin(123)\varpi_{43},$$
$$m\cos(\rho 4) = \sin(234)\varpi_{41} + \sin(134)\varpi_{42} + \sin(124)\varpi_{43} + \sin(123)\varpi_{44}.$$

These formulæ determine the only possible translation which can be effected by twists about four given screws.

As θ_1, θ_2, θ_3, θ_4 have no resultant force, then, whatever ρ may be, we must have

$$\sin(234)\cos(\rho 1) + \sin(134)\cos(\rho 2) + \sin(124)\cos(\rho 3) + \sin(123)\cos(\rho 4) = 0.$$

If in this we substitute for $\cos(\rho 1)$, &c., from the above, we have

$$\begin{aligned}&\sin^2(234)p_1 + \sin^2(134)p_2 + \sin^2(124)p_3 + \sin^2(123)p_4\\ &+ 2\sin(234)\sin(134)\varpi_{12} + 2\sin(234)\sin(124)\varpi_{13} + 2\sin(234)\sin(123)\varpi_{14}\\ &+ 2\sin(134)\sin(124)\varpi_{23} + 2\sin(134)\sin(123)\varpi_{24} + 2\sin(124)\sin(123)\varpi_{34} = 0.\end{aligned}$$

Another consequence of these formulæ may be noted. The only screw of infinite pitch in a four-system is parallel to the axis of the reciprocal cylindroid. Hence ρ is parallel to the axis of the reciprocal cylindroid. We thus have in the values of $\cos(\rho 1)$, $\cos(\rho 2)$, $\cos(\rho 3)$, $\cos(\rho 4)$, just given, the cosines of the angles between the axis of the cylindroid and the four screws 1, 2, 3, 4.

§ 25. Let θ, ϕ be any two reciprocal screws in a two-system, and let λ, μ be any two screws whatever. Then, so long as λ and μ remain unchanged, the expression

$$\frac{\cos(\lambda\theta)\cos(\mu\theta)}{p_\theta} + \frac{\cos(\lambda\phi)\cos(\mu\phi)}{p_\phi}$$

will not alter when for θ and ϕ we introduce different pairs of reciprocals

belonging to the same two-system. This follows from the general formula (Treatise, p. 288), which we may write

$$\frac{\cos(a1)\cos(b1)}{p_1}+\frac{\cos(a2)\cos(b2)}{p_2}=-\frac{\cos(a3)\cos(b3)}{p_3}-\ldots-\frac{\cos(a6)\cos(b6)}{p_6}.$$

The screws 1 and 2 are on the cylindroid reciprocal to 3, 4, 5, 6, and may be *any* pair of reciprocals on this particular cylindroid. The right-hand member is unaffected by the change of 1 and 2 into any other pair of reciprocals on the same cylindroid; therefore the same change must not affect the left-hand member, which proves the theorem.

As λ and μ may be chosen arbitrarily, we can deduce the fundamental properties of the cylindroid by various special assumptions for λ or μ. For example: let θ and ϕ, and θ' and ϕ', be two pairs of coreciprocals. Let λ be perpendicular both to θ and ϕ, and let μ be perpendicular to ϕ' but not to θ', then the equation

$$\frac{\cos(\lambda\theta)\cos(\mu\theta)}{p_\theta}+\frac{\cos(\lambda\phi)\cos(\mu\phi)}{p_\phi}=\text{const.},$$

by substitution of zero for $\cos(\lambda\theta)$ and $\cos(\lambda\phi)$, shows that the constant $=0$ (for the case of $p_\theta=0$ or $p_\phi=0$ need not be considered), whence

$$\frac{\cos(\lambda\theta')\cos(\mu\theta')}{p_\theta'}+\frac{\cos(\lambda\phi')\cos(\mu\phi')}{p_\phi'}=0;$$

but $\cos(\mu\phi')=0$, and $\cos(\mu\theta')$ is *not* zero, so that $\cos(\lambda\theta')=0$. Thus we prove that θ' is perpendicular to λ. The same applies to any screw of the two-system, and consequently we prove that all are parallel to a plane, as of course they must be, from the fact that they are on generators of the cylindroid. Let λ, μ be now chosen parallel to this plane, and let the directions of $\lambda\,\mu$, $\theta\,\phi$, $\theta'\,\phi'$ be defined by the angles λ_0, μ_0, θ_0, ϕ_0, θ'_0, ϕ'_0 which they make with some arbitrary directions in the plane to which all are parallel. The constant function shows that

$$\frac{\cos(\lambda_0-\theta_0)\cos(\mu_0-\theta_0)}{p_\theta}+\frac{\cos(\lambda_0-\phi_0)\cos(\mu_0-\phi_0)}{p_\phi}$$
$$=\frac{\cos(\lambda_0-\theta'_0)\cos(\mu_0-\theta'_0)}{p_\theta'}+\frac{\cos(\lambda_0-\phi'_0)\cos(\mu_0-\phi'_0)}{p_\phi'}=\text{const.}$$

As this has to be true for *all* values of λ_0 and μ_0, we must have the following formulæ:—

Let $\lambda_0 = 0$ and $\mu_0 = 90°$, then

$$\frac{\sin 2\theta_0}{p_\theta} + \frac{\sin 2\phi_0}{p_\phi} = \text{const.} = a. \qquad \text{i.}$$

If λ_0 and μ_0 be first made zero, we get

$$\frac{\cos^2\theta_0}{p_\theta} + \frac{\cos^2\phi_0}{p_\phi} = \text{const.}$$

If λ_0 and μ_0 be each made 90°, we have

$$\frac{\sin^2\theta_0}{p_\theta} + \frac{\sin^2\phi_0}{p_\phi} = \text{const.}$$

Hence, by adding,

$$\frac{1}{p_\theta} + \frac{1}{p_\phi} = \text{const.} = c, \qquad \text{ii.}$$

and hence

$$\frac{\cos 2\theta_0}{p_\theta} + \frac{\cos 2\phi_0}{p_\phi} = b. \qquad \text{iii.}$$

If we eliminate p_ϕ and ϕ_0 from i., ii., and iii., we find

$$p_\theta = \frac{2}{a^2 + b^2 - c^2}(a \sin 2\theta_0 + b \cos 2\theta_0 - c).$$

If we now choose as the line from which θ_0 is measured that which makes $a = 0$, and if p_α and p_β be the extreme values of p_θ, we have

$$p_\theta = p_\alpha \cos^2\theta_0 + p_\beta \sin^2\theta_0.$$

This is the well-known expression for the pitch in the cylindroid, and the equation of the cylindroid can be readily obtained. Of course all these results are long familiar, but it seemed worth while to point out the whole theory of the cylindroid was comprised in the formula

$$\frac{\cos(\lambda\theta)\cos(\mu\theta)}{p_\theta} + \frac{\cos(\lambda\phi)\cos(\mu\phi)}{p_\phi} = \text{const.}$$

§ 26. If the directions of six coreciprocal screws be given, then their pitches are determined as to relative value. For, from the general formula (Treatise, p. 287), we have

$$\frac{\sin^2(\alpha\,1)}{p_1}+\frac{\sin^2(\alpha\,2)}{p_2}+\frac{\sin^2(\alpha\,3)}{p_3}+\frac{\sin^2(\alpha\,4)}{p_4}+\frac{\sin^2(\alpha\,5)}{p_5}+\frac{\sin^2(\alpha\,6)}{p_6}=0.$$

As we may take any screw for α, we choose in succession α as coincident with 1, 2, 3, 4, 5, 6, and we have six equations of the type

$$\frac{\sin^2(12)}{p_2}+\frac{\sin^2(23)}{p_3}+\frac{\sin^2(24)}{p_4}+\frac{\sin^2(25)}{p_5}+\frac{\sin^2(26)}{p_6}=0.$$

If we form the determinant

$$\begin{vmatrix} 0 & \sin^2(12) & \sin^2(13) & \sin^2(14) & \sin^2(15) & \sin^2(16) \\ \sin^2(21) & 0 & \sin^2(23) & \sin^2(24) & \sin^2(25) & \sin^2(26) \\ \sin^2(31) & \sin^2(32) & 0 & \sin^2(34) & \sin^2(35) & \sin^2(36) \\ \sin^2(41) & \sin^2(42) & \sin^2(43) & 0 & \sin^2(45) & \sin^2(46) \\ \sin^2(51) & \sin^2(52) & \sin^2(53) & \sin^2(54) & 0 & \sin^2(56) \\ \sin^2(61) & \sin^2(62) & \sin^2(63) & \sin^2(64) & \sin^2(65) & 0 \end{vmatrix}$$

then, from Spottiswoode's property of determinants, we have the following theorem.

The principal minors of the determinant are respectively proportional to the squares of the reciprocals of the corresponding pitches.

Thus we show that when the directions are known, the relative values of the pitches are given. If we assume an arbitrary linear magnitude as one of the pitches, then all the pitches are known. Further, all the distances between every pair of screws are known, for, inasmuch as each pair are reciprocal, we have

$$(p_1+p_2)\cos(12)-d_{12}\sin(12)=0,$$

so that, as p_1, p_2, $\sin(12)$, $\cos(12)$ are known, d_{12} is known also. There are other properties of the determinant at the beginning of this article, which it may be useful to set down.

Let the minors be A_{11}, A_{12}, &c., so that

$$\sin^2(12)A_{12} + \sin^2 13)A_{13} \ldots + \sin^2(16)A_{16} = 0.$$

Then we have

$$\frac{1}{p_1^2} = hA_{11}; \quad \frac{1}{p_1 p_2} = hA_{12} \text{ \&c.} \quad \frac{1}{p_2^2} = hA_{22}; \quad \frac{1}{p_2 p_3} = hA_{23} \text{ \&c.}$$

We know that the following formula holds for a set of six coreciprocals,

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} + \frac{1}{p_6} = 0.$$

If we multiply this equation by $\frac{1}{p_1}$, we have

$$A_{11} + A_{12} \ldots + A_{16} = 0;$$

similarly, of course,

$$A_{21} + A_{22} \ldots + A_{26} = 0;$$

and, in general, the sum of the minors, corresponding to the elements of any row, or any column, is equal to zero.

There are also other formulæ to be derived from the result of § 2, for example,

$$\frac{\sin^2(12)}{p_1 p_2} + \frac{\sin^2(13)}{p_1 p_3} + \frac{\sin^2(23)}{p_2 p_3} = \frac{\sin^2(45)}{p_4 p_5} + \frac{\sin^2(46)}{p_4 p_6} + \frac{\sin^2(56)}{p_5 p_6};$$

which shows that

$$A_{12}\sin^2(12) + A_{13}\sin^2(13) + A_{23}\sin^2(23) = A_{45}\sin^2(45) + A_{46}\sin^2(46) + A_{56}\sin^2(56).$$

In like manner

$$A_{56}\sin^2(56) = A_{12}\sin^2(12) + A_{13}\sin^2(13) + A_{14}\sin^2(14) + A_{23}\sin^2(23) + A_{24}\sin^2(24) + A_{34}\sin^2(34).$$

We can also prove that the relative values of the pitches depend upon the directions, in the following manner. We have (§ 4)

$$\frac{\sin^2(143)}{p_1 p_4 p_3} = -\frac{\sin^2(526)}{p_2 p_5 p_6},$$

$$\frac{\sin^2(234)}{p_2 p_3 p_4} = -\frac{\sin^2(156)}{p_1 p_5 p_6},$$

whence, by division,

$$\frac{p_1^2}{\sin^2(156)\sin^2(134)} = \frac{p_2^2}{\sin^2(234)\sin^2(256)};$$

a like formula holds for every other pair of pitches, and thus their relative values are expressed in terms of the directions.

A different method of proving the formula, just obtained, is derived from the fundamental principle so often referred to (Treatise, p. 288), where

$$\frac{\cos(\alpha 1)\cos(\beta 1)}{p_1} + \frac{\cos(\alpha 2)\cos(\beta 2)}{p_2} + \ldots + \frac{\cos(\alpha 6)\cos(\beta 6)}{p_6} = 0,$$

whatever screws are taken for α and β.

Let us choose for α a screw perpendicular both to 3 and 4, and for β a screw perpendicular both to 5 and 6. The formula then reduces to

$$\frac{\cos(\alpha 1)\cos(\beta 1)}{p_1} + \frac{\cos(\alpha 2)\cos(\beta 2)}{p_2} = 0.$$

But, from the properties of solid angles, we have

$$\cos(\alpha 1)\sin(34) = \sin(134); \qquad \cos(\alpha 2)\sin(34) = \sin(234);$$
$$\cos(\beta 1)\sin(56) = \sin(156); \qquad \cos(\beta 2)\sin(56) = \sin(256);$$

whence the result

$$\frac{\sin(156)\sin(134)}{p_1} + \frac{\sin(234)\sin(256)}{p_2} = 0.$$

We might indeed have derived in this manner the fundamental formula of § 5. For we have by like reasoning,

$$\frac{\sin(123)\sin(156)}{p_1} = -\frac{\sin(234)\sin(456)}{p_4},$$
$$\frac{\sin(123)\sin(246)}{p_2} = -\frac{\sin(135)\sin(456)}{p_5},$$
$$\frac{\sin(135)\sin(234)}{p_3} = -\frac{\sin(156)\sin(246)}{p_6},$$

whence, multiplying the right- and left-hand sides respectively, and cancelling,

$$\frac{\sin^2(123)}{p_1 p_2 p_3} = -\frac{\sin^2(456)}{p_4 p_5 p_6}.$$

§ 27. The six screws of a coreciprocal system are parallel to six generators of a quadric cone.

This instructive theorem was communicated to me by Mr. J. H. Grace, to whom I have had elsewhere to make acknowledgments. It is the equivalent of the well-known theorem in conics, that the six vertices of two self-conjugate triangles, with regard to a conic, also lie on a conic.

The demonstration appears at once from the formula:—

$$A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16} = 0,$$

proved in the last article. For, if we substitute the direction cosines of the angles of the rays for the expressions $\sin^2(12)$, &c., we see that A_{11} does not contain the direction cosines of the screw 1, but that these direction cosines enter in the second degree into each of the quantities A_{12}, A_{13}, &c. If therefore we multiply A_{11} by the sum of the squares of the direction cosines of 1, we have a homogeneous equation in the second degree. Thus we get a quadric cone to some generator of which 1 must be parallel, and as this cone passes through generators parallel to 2, 3, 4, 5, 6, the theorem is proved.

We can obtain the same result otherwise in this manner. We have, from the formula of the last article,

$$\frac{\sin(134)\sin(156)}{p_1} = -\frac{\sin(234)\sin(256)}{p_2},$$

$$\frac{\sin(135)\sin(146)}{p_1} = -\frac{\sin(235)\sin(246)}{p_2},$$

whence, eliminating p_1 and p_2, we have

$$\frac{\sin(134)\sin(156)\sin(235)\sin(246)}{\sin(135)\sin(146)\sin(234)\sin(256)} = 1.$$

It is of course known that, when this condition holds, the six lines

1, 2, 3, 4, 5, 6 are parallel to the generators of a quadric cone, and the required theorem, as to the six screws of a coreciprocal system, is obtained.

The following proof may be also noted. Let x_2, y_2, z_2 be the direction cosines of the line parallel to the screw 2, and x_3, y_3, z_3 those parallel to the screw 3, and so on. Write the equation—

$$\begin{vmatrix} x & y & z \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \times \begin{vmatrix} x & y & z \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{vmatrix} \times \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_5 & y_5 & z_5 \end{vmatrix} \times \begin{vmatrix} x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \\ x_6 & y_6 & z_6 \end{vmatrix}$$

$$= \begin{vmatrix} x & y & z \\ x_3 & y_3 & z_3 \\ x_5 & y_5 & z_5 \end{vmatrix} \times \begin{vmatrix} x & y & z \\ x_4 & y_4 & z_4 \\ x_6 & y_6 & z_6 \end{vmatrix} \times \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \times \begin{vmatrix} x_2 & y_2 & z_2 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{vmatrix}.$$

This obviously represents a quadric cone, and as it is satisfied by making x, y, z into x_2, y_2, z_2 respectively, this cone passes through the line parallel to 2. In like manner it passes through the lines parallel to 3, 4, 5, 6. We have therefore in this the equation of the cone through the five lines parallel to 2, 3, 4, 5, 6.

If we now observe that $\sin(134)$ is proportional to

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix},$$

the required result is at once obtained.

In view of what is to follow, it may be convenient to note also the following proof.

It is easy to show that if three straight lines, OA, OB, OC, be cut by a plane ABC, then

$$\sin(OA,\ OB,\ OC) = 2\,\frac{\text{Area } ABC \times \text{Perp from } O \text{ on } ABC}{OA \cdot OB \cdot OC}.$$

If, therefore, A, B, C, D, E, F be the six points on which the straight

lines from O, parallel to 1, 2, 3, 4, 5, 6, cut a plane, then we have

$$\frac{\sin(134)\sin(156)\sin(235)\sin(246)}{\sin(135)\sin(146)\sin(234)\sin(256)}$$
$$=\frac{ACD\cdot AEF\cdot BCE\cdot BDF}{ACE\cdot ADF\cdot BCD\cdot BEF}.$$

Let us now suppose that the section of the cone we have taken is one of the circular sections. We then make use of the property that the areas of different triangles inscribed in the same circle are proportional to the continued product of their respective sides, and so we write $AC\cdot CD\cdot AD$ instead of ACD. And similarly for the other areas. The whole expression then, obviously, reduces to unity, and the theorem has been proved.

§ 28. Symmetrical expressions for the cubes of the pitches of the six screws of a coreciprocal system.

From § 25 we have

$$\frac{\sin^2(134)\sin^2(156)}{p_1^2}=\frac{\sin^2(234)\sin^2(256)}{p_2^2},$$
$$\frac{\sin^2(136)\sin^2(145)}{p_1^2}=\frac{\sin^2(236)\sin^2(254)}{p_2^2};$$
$$\frac{\sin^2(146)\sin^2(135)}{p_1^2}=\frac{\sin^2(235)\sin^2(246)}{p_2^2}.$$

We multiply the three left-hand sides of these equations together and equate the product to that of three right-hand sides, and further apply to each side the factor

$$\sin^2(345)\sin^2(346)\sin^2(356)\sin^2(456);$$

then, taking the square roots and omitting for the moment the ambiguity of sign which will be subsequently removed, we have the following theorem.

The cube of the reciprocal of the pitch of each screw of a coreciprocal system is proportional to the continued product of the ten sines of the solid angles which can be formed from the arrangement

in threes, of all the five other screws of the system. If we multiply all such expressions by the reciprocal of the product of the twenty sines of the solid angles formed by taking every combination of them among the six, we have the theorem, which may be thus expressed:—

$$p_1^3 \propto \sin(134)\sin(135)\sin(136)\sin(156)\sin(145)\sin(146) \times \sin(123)\sin(124)\sin(125)\sin(126),$$

with symmetrical expressions for the cubes of each of the other pitches.

We have thus obtained symmetrical expressions for the cube of the pitch, and for the reciprocal of the square of the pitch; the product gives a symmetrical expression for the pitch. Thus the pitch p_1 is proportional to

$$\sin(134)\sin(135)\sin(136)\sin(156)\sin(145) \times \sin(146)\sin(123)\sin(124)\sin(125)\sin(126)$$

$$\times \begin{vmatrix} 0 & \sin^2(23) & \sin^2(24) & \sin^2(25) & \sin^2(26) \\ \sin^2(32) & 0 & \sin^2(34) & \sin^2(35) & \sin^2(36) \\ \sin^2(42) & \sin^2(43) & 0 & \sin^2(45) & \sin^2(46) \\ \sin^2(52) & \sin^2(53) & \sin^2(54) & 0 & \sin^2(56) \\ \sin^2(62) & \sin^2(63) & \sin^2(64) & \sin^2(65) & 0 \end{vmatrix}.$$

§ 29. On the character of the fifteen constants which determine a coreciprocal system.

As the six screws are parallel to the generators of a quadric cone, their directions will be specified by eleven data: five of them will determine the cone, and six more will indicate the six particular generators. We have also seen how we can express the relative values of the pitches in terms of the directions. One more datum, making twelve in all, will therefore determine all the directions, all the pitches, and all the distances of the screws forming the system. Everything indeed as to the mutual relations of the screws, and the aspect of the coreciprocal system, will be completely fixed by these twelve data. The only thing not fixed is the absolute position of the system in space. For this three more data are of course required, thus making fifteen in all.

§ 30. A geometrical theorem, which will be presently seen to have a direct application to our subject, may be here given.

Let $A_1 \ldots A_6$ be six points following each other at any distances, but in consecutive order, round the circumference of a circle. If parallel forces be applied at their six points, the intensity of the force, in each case, being inversely as the continued product of the distances of its point of application from the other five points, and if the forces at A_1, A_3, A_5 be taken in one sense, and those at A_2, A_4, A_6 in the opposite sense, then the six forces equilibrate.

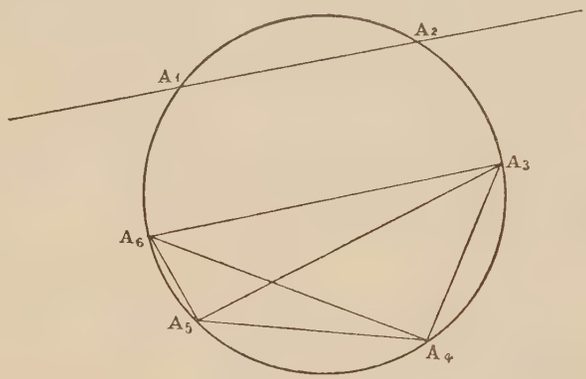

FIG. 2.

Draw the chord $A_1 A_2$ through any pair of the points, then we shall show that the moments of the forces at A_3, A_4, A_5, A_6 about $A_1 A_2$, destroy each other.

Observing that the areas of the different triangles inscribed in a circle are proportional to the continued products of their sides, we may represent the area of A_3, A_4, A_5, A_6 by either

$$A_5A_6 \cdot A_5A_4 \cdot A_6A_4 + A_3A_4 \cdot A_3A_6 \cdot A_4A_6,$$

or

$$A_4A_3 \cdot A_5A_3 \cdot A_4A_5 + A_6A_5 \cdot A_6A_3 \cdot A_5A_3.$$

Dividing by $A_3A_4 \cdot A_4A_5 \cdot A_5A_6 \cdot A_6A_3 \cdot A_3A_5 \cdot A_6A_4$, we have

$$\frac{1}{A_3A_4 \cdot A_3A_6 \cdot A_3A_5} + \frac{1}{A_4A_5 \cdot A_5A_6 \cdot A_3A_5} = \frac{1}{A_4A_6 \cdot A_5A_6 \cdot A_3A_6} + \frac{1}{A_3A_4 \cdot A_4A_5 \cdot A_4A_6},$$

which may be written in the form—

$$\frac{A_3A_2 \cdot A_3A_1}{A_3A_1 \cdot A_1A_2 \cdot A_3A_4 \cdot A_3A_5 \cdot A_3A_6} + \frac{A_5A_1 \cdot A_5A_2}{A_5A_1 \cdot A_5A_2 \cdot A_5A_3 \cdot A_5A_4 \cdot A_5A_6}$$

$$= \frac{A_6A_1 \cdot A_6A_2}{A_6A_1 \cdot A_6A_2 \cdot A_6A_3 \cdot A_6A_4 \cdot A_6A_5} + \frac{A_4A_1 \cdot A_4A_2}{A_4A_1 \cdot A_4A_2 \cdot A_4A_3 \cdot A_4A_5 \cdot A_4A_6};$$

but $A_3A_2 \cdot A_3A_1$ is to $A_5A_2 \cdot A_5A_1$ as the perpendiculars from A_3 and A_5

respectively on A_1A_2. Hence the first term in this equation is proportional to the moment of the force $1 \div A_3A_1 \cdot A_1A_2 \cdot A_3A_4 \cdot A_3A_5 \cdot A_3A_6$ about A_1A_2. Thus we prove that the sum of the moments of the forces at A_3 and A_5, about A_1A_2, is equal to the sum of the moments of the forces at A_2 and A_4. As the forces acting at the odd points are in one direction, and those on the even points in the opposite direction, it follows that they neutralize. In like manner the six forces will have no resulting moment about A_2A_3 or A_3A_4. But in this case the six forces are in equilibrium, and the theorem is proved.

If $m_1, m_2 \ldots m_n$ be a number of masses at the points $A_1 \ldots A_n$, situated anywhere, and if G be the centre of gravity of these masses, then for any point O,

$$\Sigma m_1 \, OA_1^2 = O\,G^2 \Sigma m_1 + \Sigma \, G\,A_1^2 \, m_1,$$

from which it is obvious that, if $m_1 \ldots m_6$ be the forces we have been considering, we must have

$$\Sigma m_1 \, OA_1^2 = 0;$$

whence we obtain the following geometrical theorem.

Let $A_1, \ldots A_6$ be six points ranged in this order, but at any distances apart, round the circumference of a circle, and let O be any point either in the plane of the circle or not. Then we have the following proposition :—

$$\frac{OA_1^2}{A_1A_2 \cdot A_1A_3 \cdot A_1A_4 \cdot A_1A_5 \cdot A_1A_6} + \frac{OA_3^2}{A_3A_1 \cdot A_3A_2 \cdot A_3A_4 \cdot A_3A_5 \cdot A_3A_6} + \frac{OA_5^2}{A_5A_1 \cdot A_5A_2 \cdot A_5A_3 \cdot A_5A_4 \cdot A_5A_6}$$

$$= \frac{OA_2^2}{A_2A_1 \cdot A_2A_3 \cdot A_2A_4 \cdot A_2A_5 \cdot A_2A_6} + \frac{OA_4^2}{A_4A_1 \cdot A_4A_2 \cdot A_4A_3 \cdot A_4A_5 \cdot A_4A_6} + \frac{OA_6^2}{A_6A_1 \cdot A_6A_2 \cdot A_6A_3 \cdot A_6A_4 \cdot A_6A_5}.$$

The same theorem is extended to any cyclic polygon of $2n$ sides ($n > 1$) in *Mathematical Gazette*, vol. II., No. 26.

§ 31. If from any point O, lines be drawn parallel to the six screws of a reciprocal system, and if a circular section of the quadric cone, on which these rays lie, be cut by the several rays in the points A_1, A_2, A_3, A_4, A_5, A_6, ranged in this order, then the pitches of the coreciprocals are respectively proportional to the quantities:—

$$+\frac{A_1A_2\cdot A_1A_3\cdot A_1A_4\cdot A_1A_5\cdot A_1A_6}{OA_1^2},\quad -\frac{A_2A_1\cdot A_2A_3\cdot A_2A_4\cdot A_2A_5\cdot A_2A_6}{OA_2^2},$$

$$+\frac{A_3A_1\cdot A_3A_2\cdot A_3A_4\cdot A_3A_5\cdot A_3A_6}{OA_3^2},\quad -\frac{A_4A_1\cdot A_4A_2\cdot A_4A_3\cdot A_4A_5\cdot A_4A_6}{OA_4^2},$$

$$+\frac{A_5A_1\cdot A_5A_2\cdot A_5A_3\cdot A_5A_4\cdot A_5A_6}{OA_5^2},\quad -\frac{A_6A_1\cdot A_6A_2\cdot A_6A_3\cdot A_6A_4\cdot A_6A_5}{OA_6^2}.$$

So far as the magnitudes are concerned, the proof is as follows.

We know that

$$\frac{\sin(164)\sin(135)}{p_1}=\pm\frac{\sin(264)\sin(235)}{p_2};$$

multiply each side by

$$OA_1^2\cdot OA_2^2\cdot OA_3\cdot OA_4\cdot OA_5\cdot OA_6,$$

and we have

$$\frac{OA_1\cdot OA_6\cdot OA_4\cdot\sin(164)\cdot OA_1\cdot OA_3\cdot OA_5\sin(135)\,OA_2^2}{p_1}$$
$$=\pm\frac{OA_2\cdot OA_6\cdot OA_4\sin(264)\cdot OA_2\cdot OA_3\cdot OA_5\sin(235)\,OA_1^2}{p_2};$$

but $OA_1\cdot OA_6\cdot OA_4\sin(164)$ is the volume of the parallelepiped, of which OA_1, OA_6, and OA_4 are conterminous edges, and is proportional to the product of the perpendicular from O on the plane of section into the area of the triangle $A_1A_6A_4$. But, from the property of the circle, this is proportional to $A_1A_6\cdot A_1A_4\cdot A_6A_4$, whence we have

$$\frac{p_1\,OA_1^2}{A_1A_2\cdot A_1A_3\cdot A_1A_4\cdot A_1A_5\cdot A_1A_6}=\pm\frac{p_2\,OA_2^2}{A_2A_1\cdot A_2A_3\cdot A_2A_4\cdot A_2A_5\cdot A_2A_6}.$$

Symmetry shows that the equality extends to like functions of the other four screws, and consequently the theorem is proved.

We know that the sum of the reciprocals of the pitches are equal

to zero. We also know that three of the pitches are positive and three negative. These statements must be equivalent to the formula in the last article, unless in the cases where one of the positive pitches is equal in magnitude to one of the negative pitches. Hence we see that, in general, the positive and negative pitches alternate for consecutive points on the circle.

§ 32. Through any point O let rays be drawn parallel to the six screws of a coreciprocal system. Let a plane cut these rays in points $A_1, \ldots A_6$. Then these points lie on a conic section which we shall take to be an ellipse, and we shall denote the eccentric angles defining the points $A_1, \ldots A_6$ by $\theta_1, \ldots \theta_6$ respectively. Then p_1, the pitch of the screw 1, is proportional to (sign apart)

$$OA_1^{-2} \times \sin\tfrac{1}{2}(\theta_1-\theta_2)\sin\tfrac{1}{2}(\theta_1-\theta_3)\sin\tfrac{1}{2}(\theta_1-\theta_4)\sin\tfrac{1}{2}(\theta_1-\theta_5)\sin\tfrac{1}{2}(\theta_1-\theta_6).$$

This is, of course, a generalization of the theorem of the last article, to which the theorem now given degrades when a circular section is taken.

The area of the triangle whose vertices are the points defined by θ_1, θ_2, θ_3 is proportional to $\sin\frac{1}{2}(\theta_1-\theta_2)\sin\frac{1}{2}(\theta_2-\theta_3)\sin\frac{1}{2}(\theta_3-\theta_1)$.

We have already shown, § 28, that

$$p_1^3 \propto \sin(123)\sin(124)\sin(125)\sin(126)\sin(134)\sin(135)\sin(136) \times \sin(145)\sin(146)\sin(156),$$

and we may replace $\sin(123)$ by the area of the triangle $A_1A_2A_3$ divided by $OA_1 \times OA_2 \times OA_3$, for the perpendicular from O to the plane of section is a factor which may obviously be omitted. Introducing the expression for the area just given, performing the same operation for each of the other solid angles, discarding the symmetrical factors, and taking the cube root, we obtain the required theorem.

It is proved, Salmon's *Conics* (1879) (p. 219): That the length of the chord A_1A_2 is $2b'\sin\frac{1}{2}(\theta_1-\theta_2)$, where b' is the length of the semi-diameter parallel to A_1A_2. If therefore (12) represent the chord of the ellipse joining A_1 and A_2, and $(12)'$ be the semi-diameter of the ellipse

parallel to A_1A_2, then the pitches of the six screws of the coreciprocal system are respectively proportional to

$$+ OA_1^{-2}\frac{(12)(13)(14)(15)(16)}{(12)'(13)'(14)'(15)'(16)'}, \quad - OA_2^{-2}\frac{(21)(23)(24)(25)(26)}{(21)'(23)'(24)'(25)'(26)'},$$

$$+ OA_3^{-2}\frac{(31)(32)(34)(35)(36)}{(31)'(32)'(34)'(35)'(36)'}, \quad - OA_4^{-2}\frac{(41)(42)(43)(45)(46)}{(41)'(42)'(43)'(45)'(46)'},$$

$$+ OA_5^{-2}\frac{(51)(52)(53)(54)(56)}{(51)'(52)'(53)'(54)'(56)'}, \quad - OA_6^{-2}\frac{(61)(62)(63)(64)(65)}{(61)'(62)'(63)'(64)'(65)'}.$$

The sum of the reciprocals of these six quantities is zero. Let fall a perpendicular OT from O on the plane of the ellipse, then $OA_1^2 = OT^2 + TA_1^2$. If we introduce this and similar expressions into the sum of the reciprocals, the terms involving $TA_1^2, \ldots TA_6^2$ must vanish, because we might have taken T for O in the general formula. We therefore find the terms multiplying OT^2 must disappear. We therefore have the following property of six points ranged in order 1, 2, 3, 4, 5, 6 round an ellipse:—

$$\frac{(12)'(13)'(14)'(15)'(16)'}{(12)(13)(14)(15)(16)} + \frac{(31)'(32)'(34)'(35)'(36)'}{(31)(32)(34)(35)(36)} + \frac{(51)'(52)'(53)'(54)'(56)'}{(51)(52)(53)(54)(56)} =$$

$$\frac{(21)'(23)'(24)'(25)'(26)'}{(21)(23)(24)(25)(26)} + \frac{(41)'(42)'(43)'(45)'(46)'}{(41)(42)(43)(45)(46)} + \frac{(61)'(62)'(63)'(64)'(65)'}{(61)(62)(63)(64)(65)}.$$

We can, of course, prove these results directly from the ellipse. It is easy to show that the area of any triangle inscribed in an ellipse is proportional to the continued product of the three sides divided by the continued product of the three parallel semi-diameters 1, 2, 3, 4, 5, 6.

Take any six points on the circumference of an ellipse, as in the case of the circle already considered, § 31, then the area of the quadrilateral 3 4 5 6 may be represented by either of the two triangles into which the two diagonals divide it. We thus have

$$\frac{(56)(54)(64)}{(56)'(54)'(64)'} + \frac{(34)(36)(46)}{(34)'(36)'(46)'} = \frac{(43)(53)(45)}{(43)'(53)'(45)'} + \frac{(65)(63)(53)}{(65)'(63)'(53)'};$$

which may be written thus

$$\frac{(31)'(32)'(34)'(35)'(36)'}{(31)(32)(34)(35)(36)} \times \frac{(31)(32)(12)}{(31)'(32)'(12)'} + \frac{(51)'(52)'(53)'(54)'(56)'}{(51)(52)(53)(54)(56)} \times \frac{(51)(52)(12)}{(51)'(52)'(12)'}$$

$$= \frac{(61)'(62)'(63)'(64)'(65)'}{(61)(62)(63)(64)(65)} \times \frac{(61)(62)(12)}{(61)'(62)'(12)'} + \frac{(41)'(42)'(43)'(45)'(46)'}{(41)(42)(43)(45)(46)} \times \frac{(41)(42)(12)}{(41)'(42)'(12)'}.$$

The expression $\frac{(31)(32)(12)}{(31)'(32)'(12)'}$ represents the area of 1 2 3, and this is proportional to the perpendicular from 3 on 12.

Let a force perpendicular to the plane of the ellipse, and with its point of application at 3, have a magnitude

$$-\frac{(31)'(32)'(34)'(35)'(36)'}{(31)(32)(34)(35)(36)}.$$

Let parallel forces of corresponding magnitudes act at 1, 2, 4, 5, 6, those at 1 and 5 being negative, and those at 2, 4, 6 being positive.

The first term in the equation above written is proportional to the moment of the force at 3 around the line 1 2. The equation shows that the moments of the negative forces of this type on 3 and 5 will equilibrate with moments of the same type from the positive forces on 4 and 6. In like manner, we can show that the moments of the negative forces at 1 and 3, and the positive forces at 2 and 6, will equilibrate round the line 4 5. Finally, taking moments round a third side, 3 4, we prove in the same way that the negative forces at 1 and 5, and the positive forces at 2 and 6, have no resultant moment. Thus we demonstrate that the six parallel forces, whose magnitudes are above indicated, equilibrate when those applied at the odd points, in going round the ellipse, have one sign, and those at the even points the opposite sign.

If O be any point whatever, then by the known properties of equilibrating parallel forces we have the following result, in which $O_1, \ldots O_6$ are the distances from O to the points 1, 2, 3, 4, 5, 6:—

$$O_1^2\frac{(12)'(13)'(14)'(15)'(16)'}{(12)(13)(14)(15)(16)}+O_3^2\frac{(31)'(32)'(34)'(35)'(36)'}{(31)(32)(34)(35)(36)}+O_5^2\frac{(51)'(52)'(53)'(54)'(56)'}{(51)(52)(53)(54)(56)}$$

$$=O_2^2\frac{(21)'(23)'(24)'(25)'(26)'}{(21)(23)(24)(25)(26)}+O_4^2\frac{(41)'(42)'(43)'(45)'(46)'}{(41)(42)(43)(45)(46)}+O_6^2\frac{(61)'(62)'(63)'(64)'(65)'}{(61)(62)(63)(64)(65)};$$

thus the statement of the well-known formula, that the sum of the reciprocals of the six pitches of the screws of a coreciprocal system is equal to zero, is here presented as a general geometrical relation between the squares of the distances of six points on an ellipse from any point in space.

§ 33. On a geometrical method of finding the pitches of the six screws of a coreciprocal system, when the directions of the six screws are known.

From any point O draw three lines OA, OB, OC, parallel to the three screws of positive pitch, and OX, OY, OZ, parallel respectively to the three screws of negative pitch. We have now to draw an ellipsoid of which O is the centre, and such that OX, OY, OZ are one triad of conjugate diameters, and OA, OB, OC another.

Take any point P, and imagine an ellipsoid E through P which fulfils the required condition. Draw through P a transversal PP_1 across OA, such that OA bisects PP_1, and that PP_1 is parallel to the plane OBC. Then it is plain that P_1 must lie on E. In like manner, a transversal PP_2 can be drawn across OB, which is bisected by OB and parallel to the plane OAC, then P_2 is on E, and in like manner a third point P_3 on E can be determined.

Draw PP_4 across OX, so that OX bisects PP_4 while it is parallel to the plane OYZ, then P_4 lies on E. Similarly, a transversal PP_5, across OY and bisected by it, and parallel to the plane OXZ, will give a fifth point on E, and in like manner a sixth point P_6 is determined.

Thus we have seven points P, P_1, P_2, P_3, P_4, P_5, P_6, of which any six, in addition to the centre, suffice to determine the quadric.

This is a pitch quadric, and the pitch of each screw of the six must be proportional to the inverse square of the parallel diameter of the pitch quadric. Thus the problem has been solved.

§ 34. A numerical example.

We here set down the numerical details of a particular system of six coreciprocal screws. They are, however, of a general type.

The coordinates are referred to a set of canonical coreciprocals (Treatise, p. 38), the pitches of these screws of reference being respectively

$$+141{\cdot}3,\quad -141{\cdot}3,\quad +100,\quad -100,\quad +17{\cdot}66,\quad -17{\cdot}66.$$

Screw.	*Coordinates.*					
(1)	$-{\cdot}2305$,	$-{\cdot}0759$,	$+{\cdot}3626$,	$-{\cdot}0457$,	0,	$+{\cdot}8977$,
(2)	$-{\cdot}2474$,	$+{\cdot}1813$,	$+{\cdot}1287$,	$+{\cdot}1317$,	0,	$+{\cdot}9633$,
(3)	0 ,	0 ,	0 ,	0 ,	1,	0 ,
(4)	$-{\cdot}2473$,	$+{\cdot}0006$,	$+{\cdot}2432$,	$-{\cdot}3506$,	0,	$+{\cdot}9631$,
(5)	$-{\cdot}4376$,	0 ,	0 ,	0 ,	0,	$+{\cdot}8992$,
(6)	$-{\cdot}2244$,	$-{\cdot}1804$,	$+{\cdot}2284$,	$+{\cdot}0412$,	0,	$+{\cdot}8737$.

Let $\theta_1 \ldots \theta_6$ be the coordinates of one of these screws, and $\phi_1 \ldots \phi_6$ be those of another. Then we have, if $\pm a$, $\pm b$, $\pm c$ be the pitches of the screws of reference whose magnitudes have been already given

$$a(\theta_1\phi_1 - \theta_2\phi_2) + b(\theta_3\phi_3 - \theta_4\phi_4) + c(\theta_5\phi_5 - \theta_6\phi_6) = 0.$$

The pitch of each of the six screws is found from the formula

$$p_\theta = a(\theta_1^2 - \theta_2^2) + b(\theta_3^2 - \theta_4^2) + c(\theta_5^2 - \theta_6^2).$$

The pitches are as follows:—

$$p_1 = +5{\cdot}398, \qquad p_2 = -12{\cdot}47,$$
$$p_3 = +17{\cdot}66, \qquad p_4 = -14{\cdot}12,$$
$$p_5 = +12{\cdot}78, \qquad p_6 = -5{\cdot}912.$$

A model can be easily constructed, the equations to the several screws being as follows:—

$$\begin{array}{lll}
(1) & x = -43{\cdot}7 - {\cdot}342\,z\ ; & y = -22{\cdot}4 + {\cdot}356\,z\,; \\
(2) & x = -\ 3{\cdot}1 - {\cdot}0686\,z\,; & y = -63{\cdot}7 + {\cdot}271\,z\,; \\
(3) & x = 0\ ; & y = 0\ ; \\
(4) & x = -60{\cdot}2 - {\cdot}257\,z\ ; & y = -40 - {\cdot}115\,z\ ; \\
(5) & x = -\ {\cdot}487\,z\ ; & y = -62{\cdot}5\ ; \\
(6) & x = -23{\cdot}2 - 464\,z\ ; & y = -\ 9{\cdot}8 + {\cdot}308\,z\,.
\end{array}$$

END OF PART XII.